Lecture 22

Complex Orthogonality

Def Two real-valued functions $\psi(x)$ and $\phi(x)$ are orthogonal over $a \le x \le b$ if

$$\int_a^b \psi(x)\,\phi(x)\,dx = 0$$

$$\langle \psi, \phi \rangle = \int_a^b \psi(x)\,\phi(x)\,dx$$

inner product of ψ and ϕ over $[a, b]$

Def Two complex-valued functions ψ and ϕ are orthogonal over $a \le x \le b$ if

$$\int_a^b \overline{\psi}(x)\,\phi(x)\,dx = 0$$

$$\langle \psi, \phi \rangle = \int_a^b \overline{\psi}(x)\,\phi(x)\,dx$$

where $\overline{\psi}(x)$ is a complex conjugate of $\psi(x)$.

Here

$$\psi(x) = f(x) + i g(x)$$

$f(x) = \mathrm{Re}\,\psi(x)$

$g(x) = \mathrm{Im}\,\psi(x)$

real-valued functions

$$\overline{\psi(x)} = f(x) - i g(x)$$

$z = x + iy$: complex number

$\bar{z} = x - iy$: complex conjugate of z

<u>Note</u>

$$\overline{e^{-i\frac{n\pi x}{L}}} = e^{i\frac{n\pi x}{L}}$$

Claim Functions $e^{-i\frac{n\pi x}{L}}$, $n=-\infty,\dots,+\infty$ are orthogonal on $x\in[-L,L]$.

$$\square \int_{-L}^{L} \overline{e^{-i\frac{m\pi x}{L}}}\cdot e^{-i\frac{n\pi x}{L}}\,dx = \int_{-L}^{L} e^{i\frac{m\pi x}{L}}\cdot e^{-i\frac{n\pi x}{L}}\,dx =$$

$$= \int_{-L}^{L} e^{i\frac{(m-n)\pi x}{L}}\,dx = \begin{cases} 2L, & m=n \\ [\dots], & m\neq n \end{cases}$$

$m\neq n$

$$[\dots] = \int_{-L}^{L} e^{i\frac{(m-n)\pi x}{L}}\,dx = \frac{L}{(m-n)\pi i} e^{i\frac{(m-n)\pi x}{L}}\Bigg|_{x=-L}^{L} =$$

$$= \frac{L}{(m-n)\pi i}\left[e^{i(m-n)\pi} - e^{-i(m-n)\pi}\right] =$$

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y

$$e^{it} - e^{-it} = \cancel{\cos t} + i\sin t - [\cancel{\cos t} - i\sin t] = 2i\sin t$$

$$= \frac{L}{(m-n)\pi i} \cdot 2i \cdot \sin \overset{0}{\cancel{(m-n)\pi}} = 0$$

$$\therefore \int_{-L}^{L} \overline{e^{-i\frac{m\pi x}{L}}} \cdot e^{-i\frac{n\pi x}{L}}\, dx = \begin{cases} 2L, & m=n \\ 0, & m \neq n \end{cases}$$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi x}{L}} \qquad \Big| \; \overline{e^{-i\frac{m\pi x}{L}}} \qquad \int_{-L}^{L}$$

$$\int_{-L}^{L} f(x) \underbrace{\overline{e^{-i\frac{m\pi x}{L}}}}_{= e^{i\frac{m\pi x}{L}}} dx \sim \sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{-L}^{L} \overline{e^{-i\frac{m\pi x}{L}}}\, e^{-i\frac{n\pi x}{L}}\, dx}_{= \begin{cases} 0, & m \neq n \\ 2L, & m = n \end{cases}}$$

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$$\therefore \quad \boxed{C_m = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\frac{m\pi x}{L}}\, dx \qquad \text{for all } m}$$

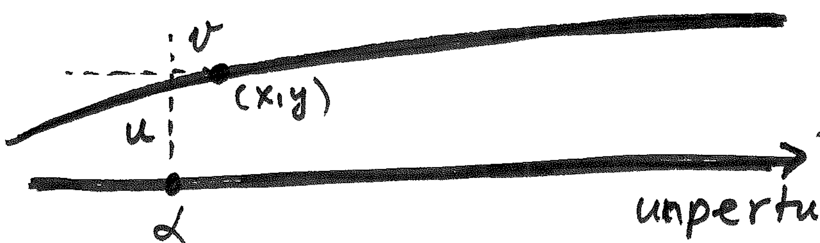
same as before (see page 9 of Lecture 21)

Complex form of a Fourier series is a more compact form of the Fourier series, and more convenient to use.

Ch 4 <u>Wave Equation</u>

<u>Vibrating String</u>

displaced/perturbed string



unperturbed string (highly stretched) in equilibrium

Let α be the x-coordinate of a particle / point sitting on a string. The string is in equilibrium, in which it is highly stretched.

Now perturb the string. Let u and v be vertical and horizontal displacements of the particle. Assume that the slope of the perturbed string is small $\Rightarrow$ we can neglect horizontal displacements and motion becomes completely vertical

$v \approx 0$

$x \approx \alpha$

$\therefore \quad y = u(x,t) \quad$ where $x = \alpha$

<u>Goal</u>: derive equation (PDE) that governs the evolution of $u(x,t)$. We will use 2[nd] Newton's Law of motion: $\vec{F} = m\vec{a}$

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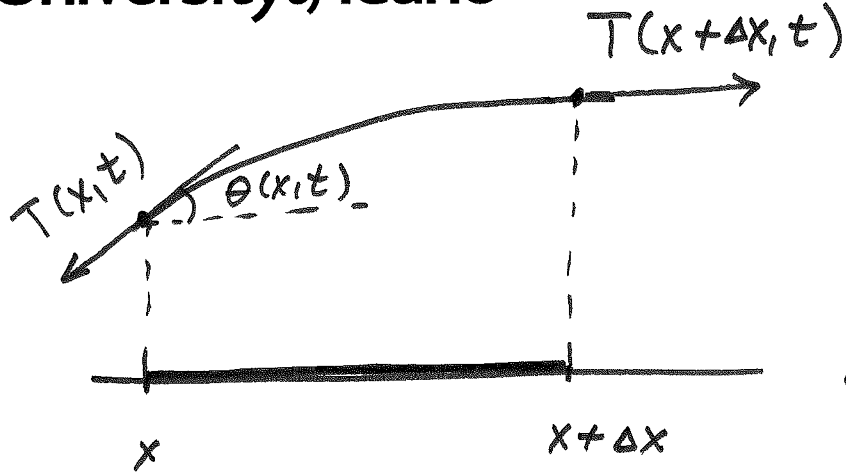

Tensile force tries to stretch the string segment.

small segment of string

Assume that mass density $\rho_0(x)$ is known. Then the total mass on the interval $[x, x+\Delta x]$ is $\rho_0(x)\Delta x$. We assume that string is <u>perfectly flexible</u> $\Rightarrow$ there is no resistance to bending $\Rightarrow$ the rest of the string exerts the force on end points that acts in the direction tangent to the string. This force is <u>tension</u> of the string. Its magnitude is $T(x,t)$. Other <u>force</u> is the body force (gravity). Denote it by $Q(x,t)$.

Let $\theta(x,t)$ be the angle between positive x-axis and tangent line to string.

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$y = u(x,t)$

Slope of string: $\frac{dy}{dx} = \tan\theta = \frac{\partial u}{\partial x}$

2nd Newton's law of motion: $m\vec{a} = \vec{F}$ (need vertical component)

$$\rho_0(x)\,\Delta x\,\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) + \rho_0(x)\,\Delta x\,\underbrace{Q(x,t)}_{\text{vertical component of body force per unit mass}}$$