Lecture 23

Wave equation (Cont'd)

Last time we derived equation

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x,t)\cdot\frac{\partial u}{\partial x}\right) + \rho_0(x)\, Q(x,t)$$

Assume that the string is <u>perfectly elastic</u> : $T(x,t) = T_0 = \text{const}$

Real strings are nearly perfectly elastic. $T(x,t)$ depends only on local stretching. Since angle $\theta(x,t)$ is small, the stretching is approximately the same everywhere, and stretching of the perturbed string is approximately the same that one of unperturbed string, which is constant.

Body force: $Q(x,t) = -g$

Assume that the body force is negligible $\Rightarrow$

$$-\rho_0(x)\, g \ll T_0 \frac{\partial^2 u}{\partial x^2}$$

$$\therefore \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

where $c^2 = \dfrac{T_0}{\rho_0(x)}$

Dimension of c:

$$\frac{\cancel{[U]}}{[t]^2} \sim [c]^2 \frac{\cancel{[U]}}{[L]^2} \quad \Rightarrow \quad [c]^2 = \frac{[L]^2}{[t]^2} = \frac{[\text{length}]^2}{[\text{time}]^2}$$

$$[c] = \frac{\text{length}}{\text{time}}$$

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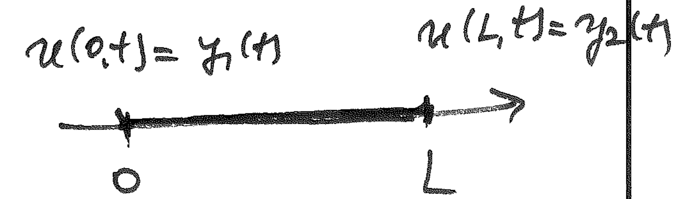
$\therefore$ c has dimension of speed.

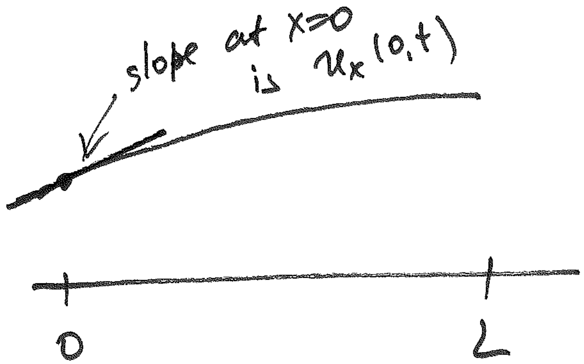
Boundary Conditions

1. Dirichlet BCs Fix string at both endpoints



$u(0,t) = y_1(t), \quad u(L,t) = y_2(t)$

2. Neumann BCs : prescribe the slope of the string at endpoints



$u_x(0,t) = y_1(t), \quad u_x(L,t) = y_2(t)$

3. <u>"Elastic" boundary condition</u>

String is attached to a mass-spring system with equilibrium position $u_E(t)$ and spring constant k.

$$T_0 \frac{\partial u}{\partial x}(0,t) = k\left(u(0,t) - u_E(t)\right)$$

this BC is homogeneous if $u_E(t) \equiv 0$ and nonhomogeneous otherwise

if $u_E(t) \equiv 0$, we have <u>homogeneous elastic BC</u>

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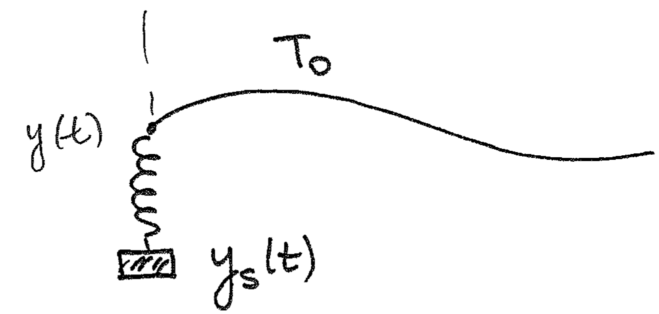

at $x=0$, $u(0,t) = y(t)$

$y(t)$ satisfies ODE for the mass-spring system

We assume that Support of mass-spring system is governed by known function $y_s(t)$.

l: length of unstretched spring, $k>0$: spring constant

$u_E(t) = y_s(t) + l$

$y(t) - y_s(t)$: length of spring

$y(t) - y_s(t) - l$: stretching of spring

Equation for vertical displacement $y(t)$:

$$m \frac{d^2 y}{dt^2} = \text{force} \qquad (2^{nd} \text{ Newton's law of motion})$$

$$m \frac{d^2 y}{dt^2} = -k\left(y(t) - y_s(t) - \ell\right) + \text{tensile force} + \text{external force}$$

Vertical component of Tension : $\quad T(0,t)\sin\theta(0,t) \approx T(0,t)\dfrac{\sin\theta(0,t)}{\cos\theta(0,t)} =$

since $\cos\theta \approx 1$ for small θ

$$= T(0,t)\underbrace{\tan\theta(0,t)}_{\text{slope of string at } x=0} = T(0,t)\frac{\partial u}{\partial x}(0,t)$$

$$m\frac{d^2 y}{dt^2} = -k\left(y(t) - y_s(t) - \ell\right) + T(0,t)\frac{\partial u}{\partial x}(0,t) + \underbrace{g(t)}_{\text{external force}}$$

Let external force (gravity) be negligible and forces are in balance (equilibrium):

$$m\frac{d^2 y}{dt^2} = 0$$

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$$\therefore \quad T(0,t)\frac{\partial u}{\partial x}(0,t) = k\left(y(t) - y_s(t) - l\right)$$

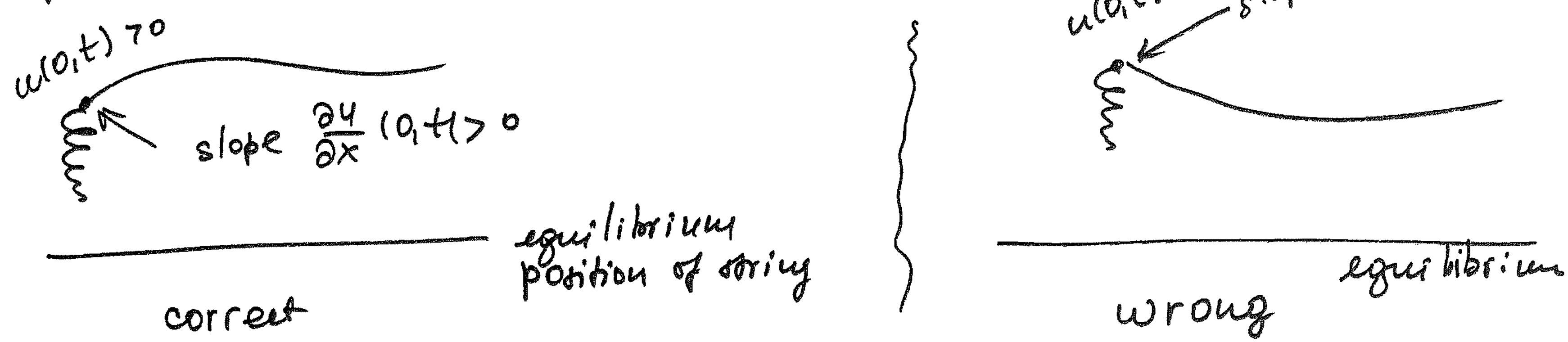
Denote $u_E(t) = y_s(t) + l$ and recall $y(t) = u(0,t)$

$$\Rightarrow \quad \boxed{T(0,t)\frac{\partial u}{\partial x}(0,t) = k\left(u(0,t) - u_E(t)\right)} \quad \text{elastic BC}$$

Elastic BC is analogue of BC for heat equation w/ Newton's law of cooling.

let $u_E(t) = 0 \Rightarrow \quad \overset{>0}{T_0}\frac{\partial u}{\partial x}(0,t) = \overset{>0}{k}\cdot u(0,t)$

if $u(0,t) > 0 \Rightarrow \frac{\partial u}{\partial x}(0,t) > 0$



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Vibrating String w/ Fixed Endpoints

$u_{tt} = c^2 u_{xx}$ on $0 < x < L$

$u(0,t) = u(L,t) = 0$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \text{IC}$$

Separation of variables: $u(x,t) = \phi(x)\, h(t)$

$\phi(x) \dfrac{d^2 h}{dt^2} = c^2 h(t) \dfrac{d^2\phi}{dx^2} \quad \Big| \quad \dfrac{1}{c^2 \phi(x) h(t)}$

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$$\underbrace{\frac{\frac{d^2h}{dt^2}}{c^2 h(t)}}_{f^n \text{ of } t} = \underbrace{\frac{\frac{d^2\phi}{dx^2}}{\phi(x)}}_{f^n \text{ of } x} = -\lambda$$

$$\frac{d^2h}{dt^2} + \lambda c^2 h = 0$$

$$\frac{d^2\phi}{dx^2} + \lambda \phi = 0$$

$$\phi(0) = \phi(L) = 0$$

<u>$h(t)$ – equation:</u>

$$\frac{d^2 h}{dt^2} + \lambda c^2 h = 0$$

$\lambda > 0 \Rightarrow h(t) = C_1 \cos\sqrt{\lambda} ct + C_2 \sin\sqrt{\lambda} ct$

$\lambda = 0 \Rightarrow h(t) = C_1 + C_2 t$

$\lambda < 0 \Rightarrow h(t) = C_1 e^{\sqrt{\lambda} ct} + C_2 e^{-\sqrt{\lambda} ct}$

Physically, we ~~expect~~ oscillations in time $\Rightarrow \boxed{\lambda > 0}$

<u>$\phi(x)$ – equation:</u> the only case when nontrivial solutions are possible is when $\lambda > 0$

$$\frac{d^2 \phi}{dx^2} + \lambda \phi = 0 \qquad \phi(0) = \phi(L) = 0 : \text{ Dirichlet BC}$$

$$\phi_n(x) = \sin\frac{n\pi x}{L} \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

$\therefore$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\frac{n\pi c t}{L}$$

IC: at $t=0$

$$\underset{f(x)}{u(x,0)} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \quad : \quad \text{Fourier sine series for } f(x)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\left(-\frac{n\pi c}{L}\right)\sin\frac{n\pi c t}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \cdot \frac{n\pi c}{L}\cos\frac{n\pi c t}{L}$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \underbrace{B_n \cdot \frac{n\pi c}{L}} \cdot \sin\frac{n\pi x}{L}$$

: Fourier sine series for $g(x)$ w/ F. coefficients

To find A_n, B_n, we use orthogonality of $\left\{ \sin \frac{n\pi x}{L} \right\}_{n=1}^{\infty}$:

$$\boxed{A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx}$$

$$B_n \cdot \frac{n\pi c}{\cancel{L}} = \frac{2}{\cancel{L}} \int_0^L g(x) \cdot \sin \frac{n\pi x}{L} \, dx$$

$$\therefore \quad \boxed{B_n = \frac{2}{n\pi c} \int_0^L g(x) \cdot \sin \frac{n\pi x}{L} \, dx}$$

Note

Heat equation: oscillatory in x, exp decay in t

Wave equation: oscillatory in both x and t

Summary: IBVP for vibrating string w/ fixed endpoints

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < L \qquad c^2 = \frac{T_0}{\rho_0}$$

$$u(0,t) = u(L,t)$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\frac{n\pi c t}{L}$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx \qquad f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\frac{n\pi x}{L}\,dx \qquad g(x) = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi c}{L} \sin\frac{n\pi x}{L}$$