Lecture 23

Last time we derived eqⁿ

$$\rho_0(x)\,\Delta x\,\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x, t)\sin\theta(x+\Delta x, t) - T(x,t)\sin\theta(x,t) + \rho_0(x)\,\Delta x\,Q(x,t)$$

Divide both sides by Δx and let $\Delta x \to 0$:

$$\rho_0(x)\,\frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x\to 0} \frac{T(x+\Delta x, t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x} + \rho_0(x)\,Q(x,t)$$

as $\Delta x \to 0$, we get

$$\rho_0(x)\,\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T(x,t)\sin\theta(x,t)\Big) + \rho_0(x)\,Q(x,t)$$

Since θ is a small angle, we have

$$\frac{\partial u}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \approx \sin\theta \quad \text{since } \cos\theta \approx 1 \text{ for small } \theta$$

Hence,

$$\boxed{\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x,t)\cdot\frac{\partial u}{\partial x}\right) + \rho_0(x)\,Q(x,t)}$$

Assume that string is <u>perfectly elastic</u> : $T(x,t) = T_0 = \text{const}$

Real strings are nearly perfectly elastic. $T(x,t)$ depends only on local stretching. Since $\theta(x,t)$ is small, the stretching is approximately the same as stretching of unperturbed string, which is constant.

Body force : $Q(x,t) = -g$

Assume that body forces are negligible :

$$-\rho_0(x)\cdot g \ll T_0\frac{\partial^2 u}{\partial x^2}$$

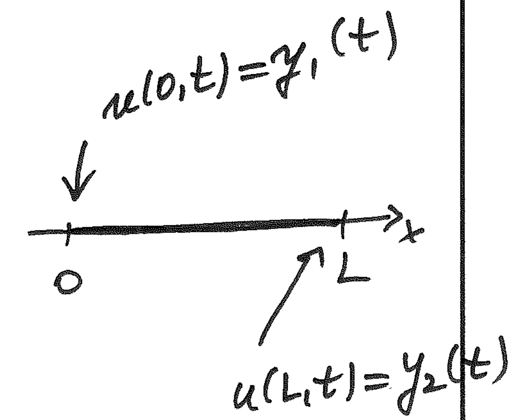
$\therefore$ $$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$ <u>wave equation</u>

where $c^2 = \dfrac{T_0}{\rho_0(x)}$

Dimension of c:

$$\frac{\overset{\text{vertical length}}{U}}{[t]^2} \sim c^2 \frac{U}{[L]^2} \Rightarrow c^2 \sim \frac{[L]^2}{[t]^2} = \left[\frac{\text{length}}{\text{time}}\right]^2$$

$\therefore$ c has dimension of velocity



Boundary conditions

1. <u>Dirichlet BC</u> Fix string at both ends

$$u(0,t) = y_1(t) \quad \text{and} \quad u(L,t) = y_2(t)$$

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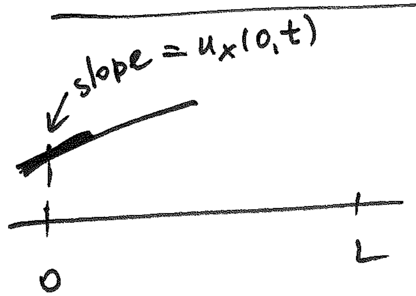
2. Neumann BC : prescribes the slope of string at endpoints



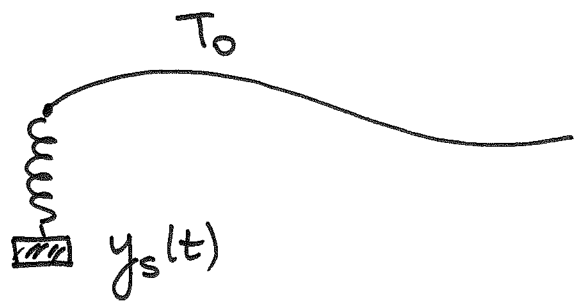
$$u_x(0,t) = y_1(t) \quad \text{and} \quad u_x(L,t) = y_2(t)$$

3. "Elastic" boundary condition

String is attached to mass-spring system with equilibrium position $u_E(t)$ and spring constant k

$$T_0 \frac{\partial u}{\partial x}(0,t) = k\left(u(0,t) - u_E(t)\right)$$: nonhomogeneous elastic BC if $u_E(t) \neq 0$

if $u_E(t) = 0$, then we say that we have homogeneous elastic BC



at $x=0$, $u(0,t) = y(t)$

$y(t)$ satisfies ODE for the mass-spring system

Support of mass-spring system is governed by known function $y_s(t)$.

l: length of unstretched spring, $k>0$: spring constant

$u_E(t) = y_s(t) + l$

$y(t) - y_s(t)$: length of spring

$y(t) - y_s(t) - l$: stretching of spring

Equation for vertical displacement $y(t)$:

$$m \frac{d^2 y}{dt^2} = \text{force}$$ (2nd Newton's law of motion)

$$m \frac{d^2y}{dt^2} = -k\left(y(t) - y_s(t) - \ell\right) + \text{tensile force} + \text{external force}$$

Vertical component of Tension : $T(0,t)\sin\theta(0,t) \approx T(0,t)\dfrac{\sin\theta(0,t)}{\cos\theta(0,t)} =$

since $\cos\theta \approx 1$ for small θ

$$= T(0,t)\underbrace{\tan\theta(0,t)}_{\text{slope of string at } x=0} = T(0,t)\frac{\partial u}{\partial x}(0,t)$$

$$m \frac{d^2y}{dt^2} = -k\left(y(t) - y_s(t) - \ell\right) + T(0,t)\frac{\partial u}{\partial x}(0,t) + \underbrace{g(t)}_{\text{external force}}$$

let external force (gravity) be negligible and forces are in balance (equilibrium):

$$m \frac{d^2y}{dt^2} = 0$$

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$$\therefore \quad T(0,t)\frac{\partial u}{\partial x}(0,t) = k\,(y(t) - y_s(t) - l)$$

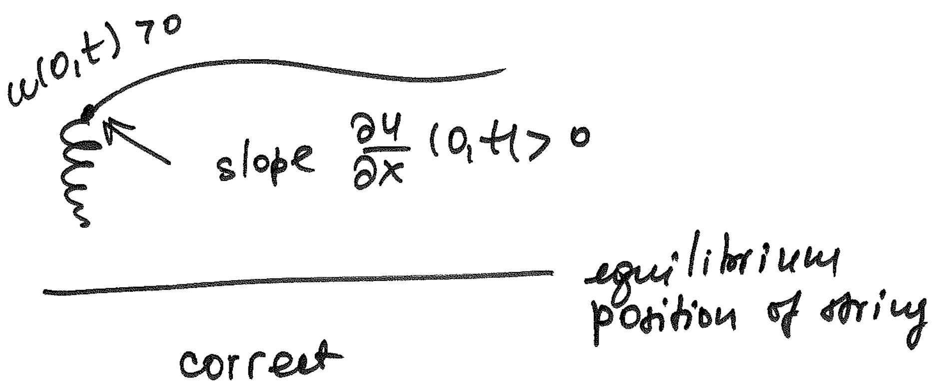
Denote $u_E(t) = y_s(t) + l$ and recall $y(t) = u(0,t)$

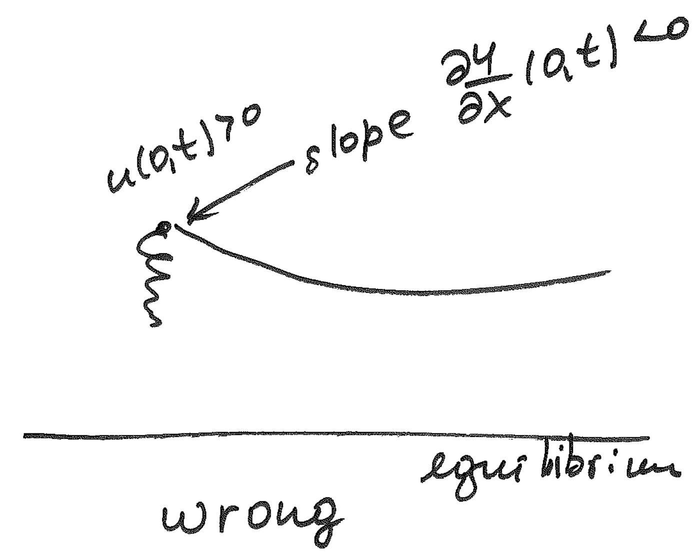
$$\Rightarrow \quad \boxed{T(0,t)\frac{\partial u}{\partial x}(0,t) = k\,(u(0,t) - u_E(t))}$$ elastic BC

Elastic BC is analogue of BC for heat equation w/ Newton's law of cooling.

let $u_E(t) = 0 \Rightarrow \quad \overset{>0}{T_0}\frac{\partial u}{\partial x}(0,t) = \overset{>0}{k}\cdot u(0,t)$

if $u(0,t) > 0 \Rightarrow \frac{\partial u}{\partial x}(0,t) > 0$





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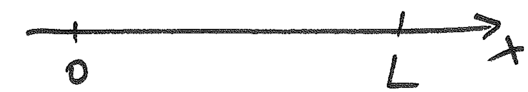
Vibrating String w/ Fixed Endpoints

$u_{tt} = c^2 u_{xx}$ on $0 < x < L$



$u(0,t) = u(L,t) = 0$ BC

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \text{IC}$$

Separation of variables: $u(x,t) = \phi(x)\, h(t)$

$$\phi(x) \frac{d^2 h}{dt^2} = c^2 \frac{d^2\phi}{dx^2} h(t) \qquad \Big| \cdot \frac{1}{c^2 \phi(x) h(t)}$$

$$\underbrace{\frac{d^2 h/dt^2}{c^2 h(t)}}_{f^{\underline{n}} \text{ of } t} = \underbrace{\frac{d^2\phi/dx^2}{\phi(x)}}_{f^{\underline{n}} \text{ of } x} = -\lambda$$

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$$\frac{d^2 h}{dt^2} + \lambda c^2 h = 0$$

$$\frac{d^2 \phi}{dx^2} + \lambda \phi = 0$$

$$\phi(0) = \phi(L) = 0$$