Lecture 24

Interpretation of the wave equation solutions in the context of musical stringed instruments

Last time we derived solution $u(x,t)$, a vertical displacement, is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi x}{L} \sin\frac{n\pi ct}{L} \qquad (1)$$

or

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right] \qquad (1')$$

i.e. $u(x,t)$ is an infinite sum of terms

$$\sin\frac{n\pi x}{L}\left[A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right] \qquad (*)$$

$$A_n \cos\frac{n\pi c t}{L} + B_n \sin\frac{n\pi c t}{L} = \sqrt{A_n^2+B_n^2}\left(\underbrace{\frac{A_n}{\sqrt{A_n^2+B_n^2}}}_{\sin\theta}\cos\frac{n\pi c t}{L} + \underbrace{\frac{B_n}{\sqrt{A_n^2+B_n^2}}}_{\cos\theta}\sin\frac{n\pi c t}{L}\right) = \underbrace{\sqrt{A_n^2+B_n^2}}_{\gamma}\sin\left(\frac{n\pi c t}{L}+\theta\right)$$

$$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$$

θ: phase shift

γ: amplitude

$$\gamma = \sqrt{A_n^2+B_n^2}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{A_n}{B_n}$$

Check:

$$\sin^2\theta + \cos^2\theta \stackrel{?}{=} 1$$

$$\left(\frac{A_n}{\sqrt{A_n^2+B_n^2}}\right)^2 + \left(\frac{B_n}{\sqrt{A_n^2+B_n^2}}\right)^2 = 1 \quad ✓$$

Amplitude $\gamma = \sqrt{A_n^2+B_n^2}$ determines intensity of the sound.

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Temporal part $A_n \cos\frac{n\pi c t}{L} + B_n \sin\frac{n\pi c t}{L}$ is a simple harmonic motion w/ angular / circular frequency $\frac{n\pi c}{L}$.

$\omega = \frac{n\pi c}{L}$: # of oscillations per 2π unit of time

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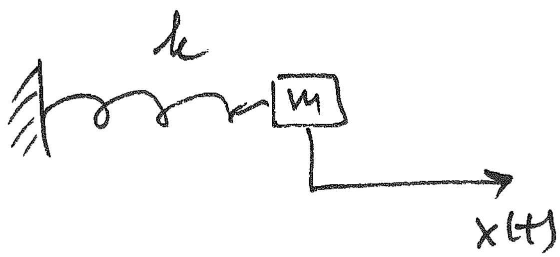
Recall mass-spring systems w/o damping



$x(t)$: displacement from equilibrium

k: spring const

m : mass

$m\ddot{x} = -kx$ or $m\ddot{x} + kx = 0$ or $\ddot{x} + \underbrace{\frac{k}{m}}_{=\omega_0^2} x = 0$

$\pm i\omega_0$

ω_0: angular frequency

$x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$: simple harmonic motion

Aside: mass-spring system w/ damping

$m\ddot{x} + c\dot{x} + kx = 0$

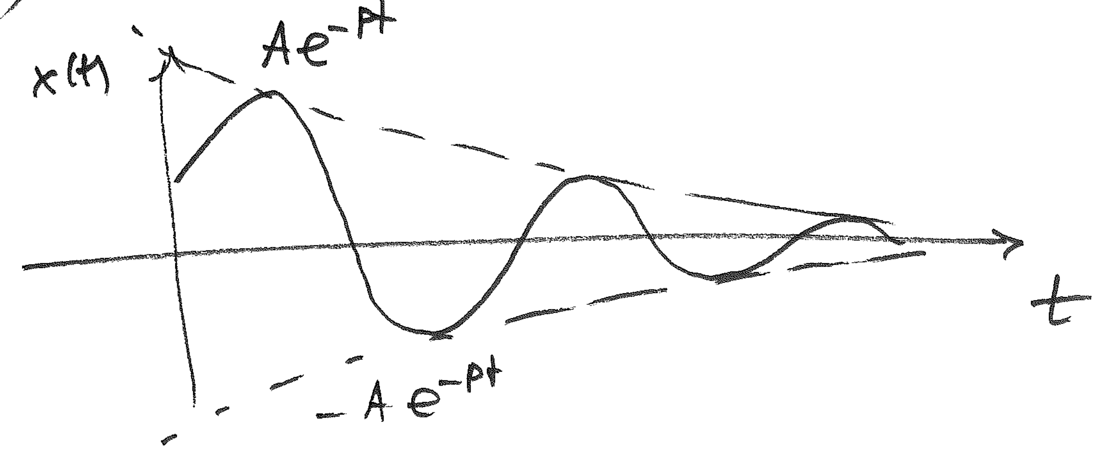
underdamped: $c^2 - 4mk < 0$ i.e. $c^2 < 4mk$

$$\frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$\frac{-c}{2m} \pm i\,\frac{\sqrt{4mk - c^2}}{2m} = -p \pm i\omega$$

$$x(t) = C_1 e^{-pt}\cos\omega t + C_2 e^{-pt}\sin\omega t = e^{-pt}\left(C_1\cos\omega t + C_2\sin\omega t\right)$$

$$= \underbrace{e^{-pt} A}_{\text{time-varying amplitude}} \cdot \cos(\omega t + \alpha) \qquad A = \sqrt{C_1^2 + C_2^2}$$



Back to wave equation.

The sound is an infinite superposition of terms (*) which are called <u>normal modes</u>. We can write them as

$$\underbrace{\sqrt{A_n^2 + B_n^2}\,\sin\left(\frac{n\pi c t}{L} + \theta\right)}_{A(t)} \cdot \sin\frac{n\pi x}{L} \qquad (**)$$

$A(t)$: time-dependent amplitude

Spatial frequency is $\frac{n\pi}{L}$, temporal frequency is $\frac{n\pi c}{L}$, which is called <u>natural frequency</u>.

n^{th} harmonics of the wave equation

When $n=1$, the natural frequency is called the first frequency or fundamental frequency

$$\omega = \frac{\pi c}{L}, \qquad c = \sqrt{\frac{T_0}{\rho_0}}$$

ω depends on T_0, ρ_0, L

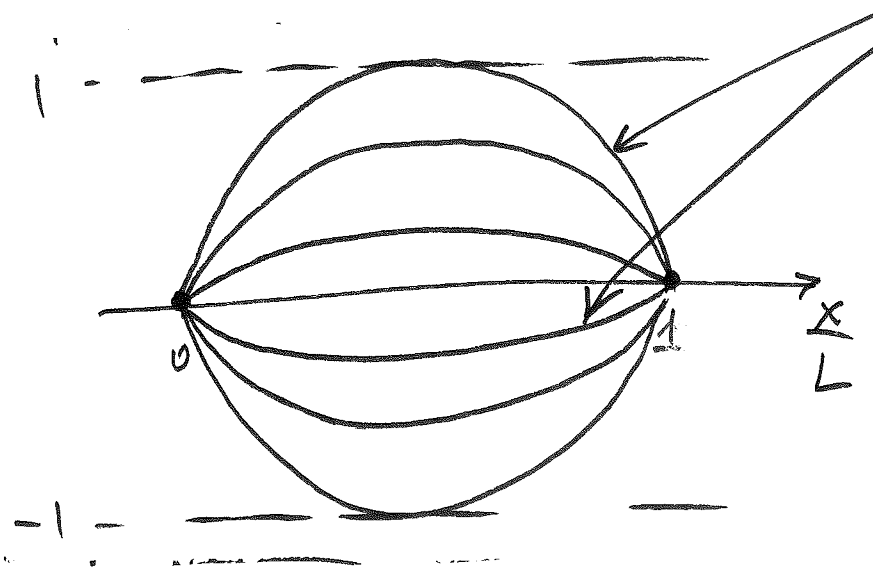
Frequency $\omega = \frac{\pi c}{L}$ can be increased by increasing c or decreasing L. Higher frequency implies higher pitch. Since density ρ_0 is constant, c can be increased by increasing tension T_0. Length can be decreased by clamping down the string.

n^{th} term in (*) or (**) is called

<u>n^{th} harmonic</u>:

$$\underbrace{\sqrt{A_n^2+B_n^2}\,\sin\left(\frac{n\pi c t}{L}+\theta\right)}_{A(t)}\cdot \sin\frac{n\pi x}{L} = A(t)\cdot\sin\frac{n\pi x}{L}$$

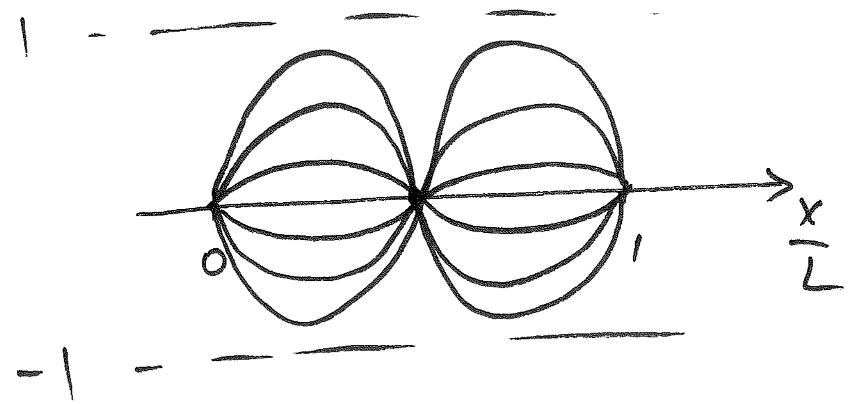
<u>$n=1$</u>: $A(t)\cdot\sin\frac{\pi x}{L}$ $\quad 0\le x\le L$



curves at different times

as time changes, the amplitude $\sin\left(\frac{n\pi c t}{L}+\theta\right)$ will vary between -1 and 1. Since there is no displacement at endpoints (due to BCs), we have <u>standing waves.</u>

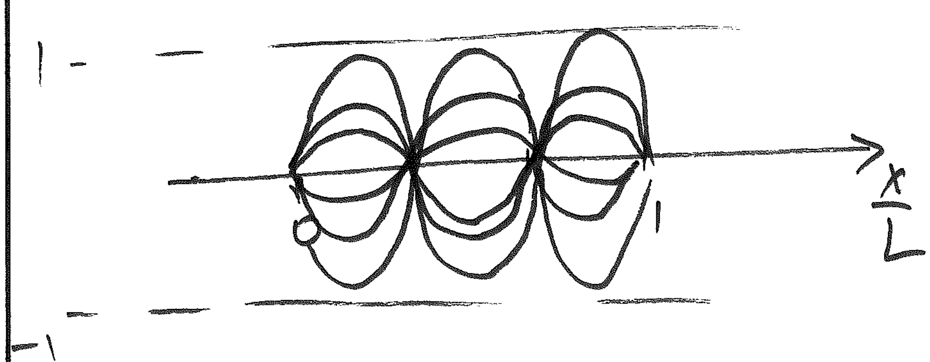
$\underline{\underline{n=2}}$: $A(t) \cdot \sin \frac{2\pi x}{L}$



Displacement $= 0$ for any time at $x = \frac{L}{2}$

2^{nd} harmonic has one node at $x = \frac{L}{2}$. <u>Node</u> <u>node</u> is a point in $(0, L)$ where displacement is zero.

$\underline{\underline{n=3}}$: $\sin \frac{3\pi x}{L}$



3^{rd} harmonic has 2 nodes at $x = \frac{L}{3}$ and $x = \frac{2L}{3}$

In general, n^{th} harmonic has $n-1$ nodes in $(0, L)$.

<u>Claim</u>: any standing wave can be decomposed into 2 traveling waves

$$\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} = \frac{1}{2}\sin\frac{n\pi}{L}(x-ct) + \frac{1}{2}\sin\frac{n\pi}{L}(x+ct)$$

$$\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L} = \frac{1}{2}\cos\frac{n\pi}{L}(x-ct) - \frac{1}{2}\cos\frac{n\pi}{L}(x+ct)$$

Solution $u(x,t)$ is superpositi~~ng~~on of standing waves

$\Rightarrow$

$$u(x,t) = \underbrace{R(x-ct)}_{\text{wave traveling to the right w/ speed } c} + \underbrace{S(x+ct)}_{\text{wave traveling to the left w/ speed } c}$$

Note: this result is also true if other BCs are used (not necessarily fixed endpoints BCs).