Lecture 24

Vibrating String w/ Fixed Endpoints (Cont'd)

$h(t)$ - equation:

$$\frac{d^2 h}{dt^2} + \lambda c^2 h = 0$$

$\lambda > 0$: $\quad h(t) = C_1 \cos\sqrt{\lambda}ct + C_2 \sin\sqrt{\lambda}ct$

$\lambda = 0$: $\quad h(t) = C_1 + C_2 t$

$\lambda < 0$: $\quad h(t) = C_1 e^{c\sqrt{\lambda}t} + C_2 e^{-c\sqrt{\lambda}t}$

Physically, we expect oscillatory solution in time $\Rightarrow \lambda > 0$.

$\phi(x)$ - equation : the only case when we have a nontrivial solution is when $\lambda > 0$

$$\frac{d^2 \phi}{dx^2} + \lambda \phi = 0$$

$\phi(0) = \phi(L) = 0$: Dirichlet BCs

$$\phi_n(x) = \sin\frac{n\pi x}{L} \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi c t}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi c t}{L}$$

IC: at $t=0$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$: Fourier sine series for $f(x)$

$$u_t(x,0) = \sum_{n=1}^{\infty} \underbrace{B_n \frac{n\pi c}{L}}_{\text{Fourier coefficients of } g(x)} \sin\frac{n\pi x}{L} = g(x)$$: Fourier sine series for $g(x)$

To find A_n, B_n, we use orthogonality of $\left\{\sin\frac{n\pi x}{L}\right\}_{n=1}^{\infty}$:

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}$$

$$B_n \cdot \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\,dx$$

$$\therefore \quad B_n = \frac{\cancel{L}}{n\pi c}\cdot\frac{2}{\cancel{L}}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$$\boxed{B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx}$$

<u>Note</u>

<u>Heat equation</u>: oscillatory in x, exp decay in t

<u>Wave equation</u>: oscillatory in both x and t

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Summary : IBVP for vibrating string w/ fixed endpoints

$$\begin{cases} u_{tt} = c^2 u_{xx} \qquad 0 < x < L \qquad c^2 = \dfrac{T_0}{\rho_0} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases}$$

Solution

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L} + B_n \sin\frac{n\pi x}{L} \sin\frac{n\pi c t}{L}} \qquad (1)$$

IC: $u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi c}{L} \sin\frac{n\pi x}{L} = g(x)$$

Interpretation in the context of musical stringed instruments

(with fixed ends)

Solution (1), i.e. vertical displacement $u(x,t)$, is written as infinite superposition (linear combination) of terms:

$$\boxed{\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L}+B_n\sin\frac{n\pi ct}{L}\right)} \qquad (*)$$

$$A_n\cos\frac{n\pi ct}{L}+B_n\sin\frac{n\pi ct}{L}=\sqrt{A_n^2+B_n^2}\left(\underbrace{\frac{A_n}{\sqrt{A_n^2+B_n^2}}}_{\sin\theta}\cos\frac{n\pi ct}{L}+\underbrace{\frac{B_n}{\sqrt{A_n^2+B_n^2}}}_{\cos\theta}\sin\frac{n\pi ct}{L}\right) =$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\cdot\sin\beta$$

$$= \underbrace{\sqrt{A_n^2+B_n^2}}_{\gamma}\sin\left(\frac{n\pi ct}{L}+\theta\right)$$

To check:

$$\sin^2\theta + \cos^2\theta = \frac{A_n^2}{A_n^2+B_n^2} + \frac{B_n^2}{A_n^2+B_n^2} = 1 \quad \checkmark$$

$\gamma \equiv \sqrt{A_n^2+B_n^2}$: amplitude

θ: phase shift

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{A_n}{B_n}$$

$$\therefore \quad A_n \cos\frac{n\pi c t}{L} + B_n \sin\frac{n\pi c t}{L} = \sqrt{A_n^2+B_n^2}\,\sin\left(\frac{n\pi c t}{L} + \theta\right)$$

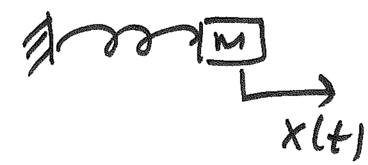
Amplitude $\gamma = \sqrt{A_n^2+B_n^2}$ determines intensity of the sound.

Temporal part of (*) is a simple harmonic motion w/ angular / circular frequency

$\omega = \frac{n\pi c}{L}$: # of oscillations per 2π units of time aside

Recall mass-spring system oscillations w/o friction

$x(t)$: displacement from equilibrium



$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \omega^2 = \frac{k}{m}$$

$$\ddot{x} + \omega^2 x = 0$$

$\pm i\omega$

$$x(t) = C_1 \cos\omega t + C_2 \sin\omega t =$$

$$= \sqrt{C_1^2 + C_2^2} \cdot \sin(\omega t + \theta)$$: simple harmonic motion

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Back to wave eq^n.

The sound is ∞ superposition of terms (*) which are called <u>normal modes</u>.

We can write (*) as

$$\underbrace{\sqrt{A_n^2 + B_n^2}\,\sin\left(\frac{n\pi ct}{L} + \theta\right)}_{A(t)} \sin\frac{n\pi x}{L} \qquad (**)$$

$A(t)$: time-dependent amplitude

Spatial frequency is $\frac{n\pi}{L}$, while temporal frequency is $\frac{n\pi c}{L}$, also called natural frequency.

When $n=1$, this frequency is called the first frequency or fundamental frequency. Other frequencies are integral multiples of the fundamental frequency.

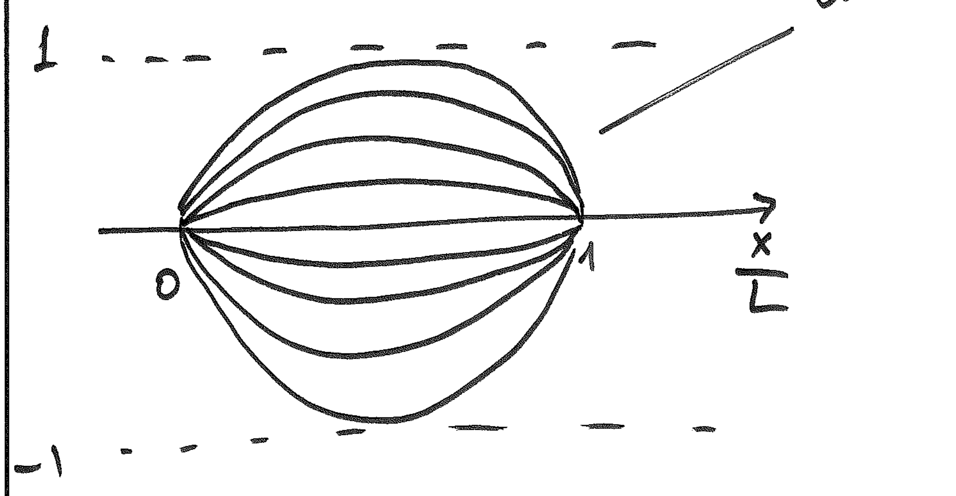
$$\omega = \frac{\pi c}{L} : \quad \text{depends on } c = \sqrt{\frac{T_0}{\rho_0}},\ L$$

Frequency $\omega = \frac{\pi c}{L}$ can be increased by increasing c or decreasing L. Higher frequency implies higher pitch. Since density ρ_0 is const, c can be changed by increasing tension T_0. Length can be decreased by clamping down the string.

n^{th} term in (*) or (**) is called n^{th} harmonic:

$$\underbrace{\sqrt{A_n^2+B_n^2}\,\sin\left(\frac{n\pi c t}{L}+\theta\right)}_{A(t)}\sin\frac{n\pi x}{L}$$

$n=1$: $\quad A(t)\sin\frac{\pi x}{L}$



as time changes, the amplitude $\sin\left(\frac{n\pi c t}{L}+\theta\right)$ will vary between -1 and 1. Since there is no displacement at end points (due to BCs), we have <u>standing waves</u>