Lecture 25

Recall solution to the wave equation

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}$$

$$\boxed{u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)} \quad \overset{(*)}{\Rightarrow} \text{ can find } A_n\text{'s}$$

$$\boxed{u_t(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\cdot\frac{n\pi c}{L} = g(x)} \quad \Rightarrow \text{ can find } B_n\text{'s}$$

Ex Let $u(x,0) = f(x)$, $u_t(x,0) = 0$, i.e. $g(x) = 0$.

Since $g(x) = 0 \Rightarrow B_n = 0 \quad n = 1, 2, \dots$

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$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L} =$$

$$= \sum_{n=1}^{\infty} \frac{A_n}{2}\left[\sin\frac{n\pi}{L}(x-ct) + \sin\frac{n\pi}{L}(x+ct)\right] =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x-ct) + \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x+ct) \overset{\text{see } (*)}{\underset{\text{for } f(x)}{=}}$$

$$= \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct)$$

$$\therefore \quad \boxed{u(x,t) = \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct)}$$

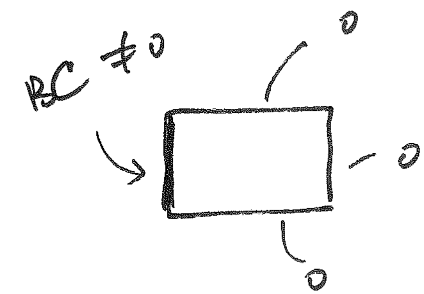
Chapter 1-4 Summary

- Solution method: separation of variables
- Considered three equations from classical physics: heat eq^n, Laplace eq^n and wave eq^n

(1D)

1. Heat eq^n: $u_t = k u_{xx}$

   - exp decay in time
   - oscillatory in space



2. Laplace equation: $\nabla^2 u = 0$

   $u_{xx} + u_{yy} = 0$ (in 2D)

- oscillatory in one variable
- non-oscillatory in another variable
- Laplace eq^n is a steady-state analogue of the heat and wave equations

$$\cancel{u_t}^{\,0} = k\nabla^2 u \Rightarrow \nabla^2 u = 0$$

$$\cancel{u_{tt}}^{\,0} = c^2\nabla^2 u \Rightarrow \nabla^2 u = 0$$

3. Wave eq^n: $u_{tt} = c^2 u_{xx}$ (1D)

- oscillatory in time and space
- solution is a sum of two travelling waves $R(x-ct)$ and $S(x+ct)$, right and left traveling waves

Note: we looked at constant coefficients cases, homogeneous problems, 2 independent variables.

Chapter 5: Sturm–Liouville theory: non-uniform material properties

Summary: Separation of variables (with homogeneous BCs & homog. DEs) gives

BVP: $\phi''(x) + \lambda\phi(x) = 0$ $\quad 0 \le x \le L$ or $-L \le x \le L$

with homogeneous BCs:

1. Dirichlet BCs: $\phi(0) = \phi(L) = 0$: Fourier sine series
2. Neumann BCs: $\phi'(0) = \phi'(L) = 0$: Fourier cosine series

3. Periodic BCs: $\phi(-L) = \phi(L)$, $\phi'(-L) = \phi'(L)$: Fourier series

<u>Key concepts:</u> linearity + homogeneity + orthogonality

<u>Q</u> What can we do if problems involve equations with non-constant coefficients?

<u>Ex</u> Heat flow in a non-uniform rod.

$$c\rho u_t = (K_0 u_x)_x + Q \quad : \quad \text{linear non-homog. eq}^{\text{n}}$$

where $c = c(x)$, $\rho = \rho(x)$, $K_0 = K_0(x)$

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Let $Q(x) = \alpha(x)\, u(x,t)$: heat source depends on temperature (linearly)

Then

$$c\rho u_t = (K_0 u_x)_x + \alpha(x) u \quad : \quad \text{linear homog. eq}^{n}$$

Separation of variables:

$$u(x,t) = \phi(x)\, h(t)$$

$$c(x)\,\rho(x)\,\phi(x)\frac{dh}{dt} = h(t)\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \alpha(x)\,\phi(x)\,h(t) \;\Big|\cdot \frac{1}{c\rho\phi h}$$

$$\underbrace{\frac{1}{h}\frac{dh}{dt}}_{f^{n} \text{ of } t} = \underbrace{\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right)\cdot\frac{1}{c(x)\rho(x)\,\phi(x)} + \alpha(x)\frac{1}{c(x)\rho(x)}}_{f^{n} \text{ of } x} = -\lambda$$

$$\boxed{\begin{aligned} &\frac{dh}{dt} + \lambda h = 0 \\ &\frac{d}{dx}\left(K_0\right)\frac{d\phi}{dx} + \alpha(x)\,\phi(x) + \lambda\, c(x)\rho(x)\,\phi(x) = 0 \end{aligned}}$$

$h(t) = Ce^{-\lambda t}$

$\phi(x)$ – ?   Cannot solve in general

Goal: understand behaviour of $\phi(x)$ without having it in a closed form