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Lecture 25

Midterm Review: tomorrow, Tuesday,
March 21, 9:30 – 10:20 am
JEB 21
Exam covers Chapters 1–3

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n^{th} harmonics of wave equation

$$\underbrace{\sqrt{A_n^2 + B_n^2}\,\sin\left(\frac{n\pi c t}{L} + \theta\right)}\,\sin\frac{n\pi x}{L}$$

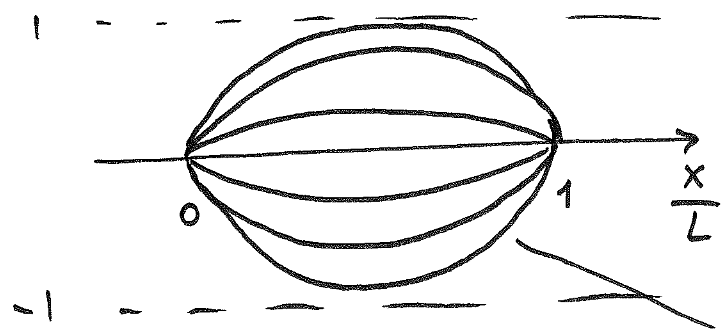
$A(t)$: time-varying amplitude

we will plot

$$\sin\left(\frac{n\pi c t}{L} + \theta\right)\sin\frac{n\pi x}{L}$$

at different times.

$n=1$



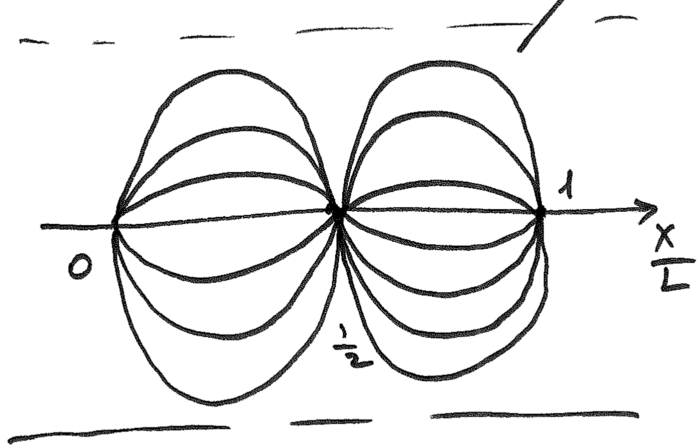
$\sin \frac{\pi x}{L}$

standing waves

curves at different times

$\sin \frac{2\pi x}{L}$

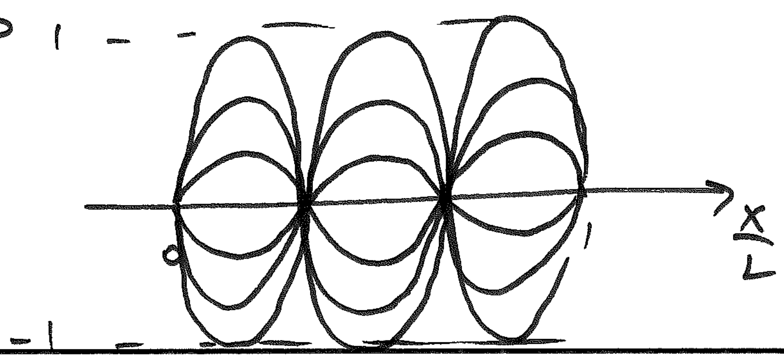
$n=2$



Displacement $= 0$ for any time at $x = \frac{L}{2}$.

This point is called a <u>node</u>

$n=3$



3^{rd} harmonic has 2 nodes at $x = \frac{L}{3}$ and $x = \frac{2L}{3}$

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In general, one can show that the n^{th} harmonic has $n-1$ nodes.

<u>Claim</u>: any standing wave can be decomposed into 2 traveling waves

$$\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} = \frac{1}{2}\sin\frac{n\pi}{L}(x-ct) + \frac{1}{2}\sin\frac{n\pi}{L}(x+ct)$$

$$\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L} = \frac{1}{2}\cos\frac{n\pi}{L}(x-ct) - \frac{1}{2}\cos\frac{n\pi}{L}(x+ct)$$

Solution $u(x,t)$ is superposition of standing waves $\Rightarrow$

$$u(x,t) = \underbrace{R(x-ct)}_{\substack{\text{wave traveling}\\ \text{to the right}\\ \text{w/ speed } c}} + \underbrace{S(x+ct)}_{\substack{\text{wave traveling}\\ \text{to the left}\\ \text{w/ speed } c}}$$

<u>Note</u>: this result is true even if BC are not fixed at endpoints (at $x=0$ and $x=L$)

Recall,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\frac{n\pi c t}{L}$$

$$\boxed{u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi c}{L} \sin\frac{n\pi x}{L} = g(x)$$

<u>Ex</u> let $u(x,0) = f(x)$, $u_t(x,0) = 0$

$\Rightarrow g(x) = 0 \Rightarrow B_n = 0$, $n = 1, 2, \ldots$

$$\therefore \; u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L} =$$

$$= \sum_{n=1}^{\infty} A_n \cdot \frac{1}{2}\left[\sin\frac{n\pi}{L}(x-ct) + \sin\frac{n\pi}{L}(x+ct)\right] =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}(x-ct) + \frac{1}{2}\sum_{n=1}^{\infty} A_n \cdot \sin\frac{n\pi}{L}(x+ct) =$$

$$= \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct)$$

$$\therefore \quad \boxed{u(x,t) = \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct)}$$

Chapter 1-4 Summary

- Solution method: separation of variables
- Considered three equations from classical physics:

1. **Heat equation:** $u_t = k u_{xx}$ (1D)

- exp decay in time
- oscillatory in space

2. **Laplace equation:** $\nabla^2 u = 0$

$u_{xx} + u_{yy} = 0$ (in 2D)

- oscillatory in one variable
- non-oscillatory in another variable
- Laplace equation is the steady-state analogue of heat equation and wave equation,

$$\cancel{u_t}^{\,0} = k\nabla^2 u \Rightarrow \nabla^2 u = 0$$

$$\cancel{u_{tt}}^{\,0} = c^2\nabla^2 u \Rightarrow \nabla^2 u = 0$$

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3. <u>Wave equation</u> : $u_{tt} = c^2 u_{xx}$ (1D)

- oscillatory in time and space
- solution is a sum of two traveling waves $R(x-ct)$ and $S(x+ct)$, right and left traveling waves.

Looked at only constant coefficients cases, homogeneous problems, & independent variables.

<u>Ch 5</u> : Sturm-Liouville theory : non-uniform material properties

<u>Ch 7</u> : Higher Dimensional Problems

<u>Ch 8</u>: Nonhomogeneous Problems

Ch 9 : Green's functions

Ch 10 : Infinite domain problems: Fourier transform method

if time allows

Ch 5 Sturm - Liouville Theory

Summary: Separation of variables

BVP: $\phi''(x) + \lambda \phi(x) = 0$ $\quad 0 \leq x \leq L$ or $-L \leq x \leq L$

with homogeneous BCs:

1. Dirichlet BCs: $\phi(0) = \phi(L) = 0$: Fourier sine series
2. Neumann BCs: $\phi'(0) = \phi'(L) = 0$: Fourier cosine series

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3. Periodic BCs: $\phi(-L) = \phi(L)$: Fourier series

$\phi'(-L) = \phi'(L)$

<u>Key concepts</u>: linearity + homogeneity + orthogonality

<u>Q</u> What can we say about non-constant coefficient problems?