Lecture 26

Midterm Review

Exam covers Chapters 1-4.

Ch 1 Heat equation

- no derivation
- steady state solution
- various boundary conditions, physical meaning
- properties of solutions (oscillatory, exp decay)

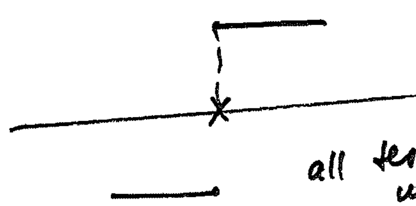
Ch 2 Separation of variables

- in Cartesian coordinates (x,y) (or in (t,x))
- in polar coordinates (r,θ)
- principle of linear superposition
- orthogonality of eigenfunctions

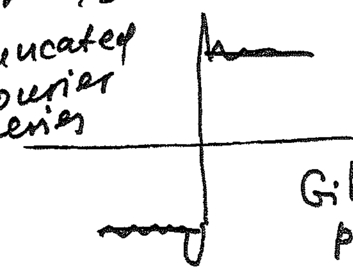
- review standard problems on separation of variables that lead to Fourier sine, Fourier cosine and Fourier series
- no fluid problem
- Laplace equation, properties of solutions (oscillatory, non-oscillatory etc.)

Ch3 Fourier series

- Fourier sine, Fourier cosine, Fourier series
- convergence of Fourier sine, cosine and Fourier series
- periodic extensions, even and odd periodic extensions
- graphs of Fourier series including graphs of truncated Fourier series, say, the first 10 or 100 terms



all terms are used



truncated Fourier series

Gibbs phenomenon

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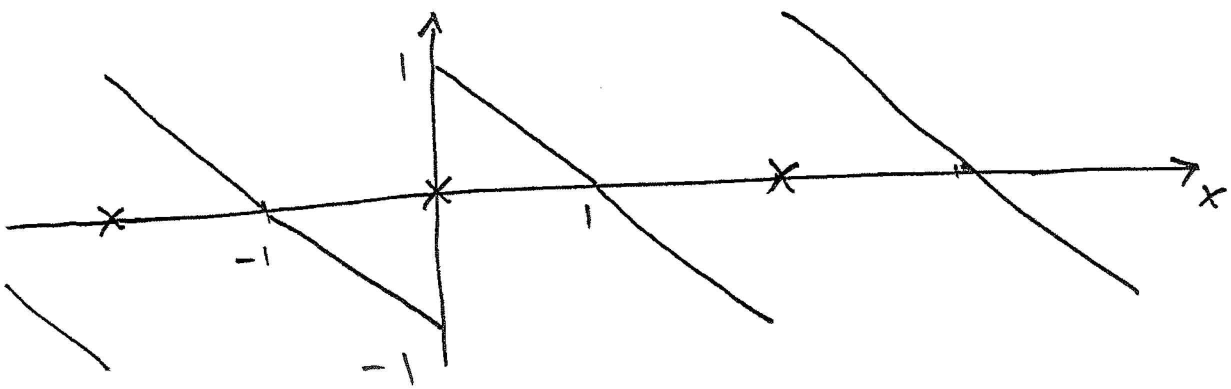
- odd and even parts of a function
- term-by-term differentiation and integration of a Fourier series
- complex form of Fourier series

Ch 4 wave equation: separation of variables, conservation of energy, modes, standing & traveling waves

You may bring one sheet, one sided, letter size, of your notes

Ex $f(x) = 1 - x, \quad 0 \le x \le 1$

Sketch Fourier sine series



graph of a Fourier sine series

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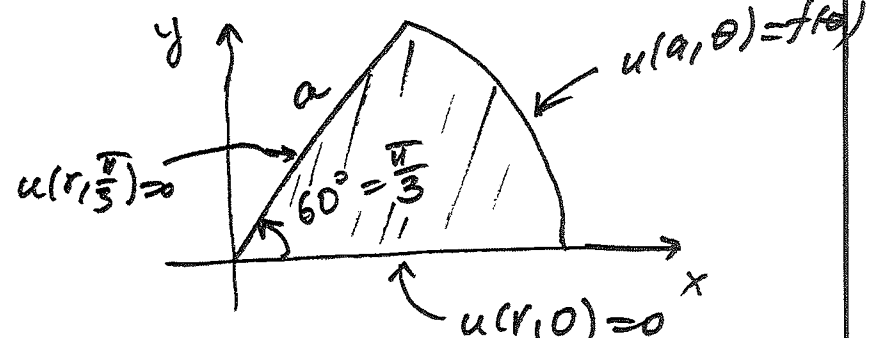
$$f_o(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

$f(x) = 1-x \qquad f(-x) = 1+x$

$$f_o(x) = \frac{1}{2}\left(1-x-(1+x)\right) = \frac{1}{2}(-2x) = -x$$

Fourier series of $f_o(x) = -x$ is Fourier sine series since $-x$ is odd and all coefficients of cosine terms $A_n = 0$.

Ex Using separation of variables, solve Laplace's equation inside a 60° wedge of radius a subject to the following boundary conditions:



$u(r,0) = 0$

$u(r,\frac{\pi}{3}) = 0$

$u(a,\theta) = f(\theta)$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

separation of variables: $u(r,\theta) = G(r)\,\psi(\theta)$

$$\nabla^2 u = 0$$

$$\psi(\theta)\,\frac{1}{r}\frac{d}{dr}\left(r\frac{dG(r)}{dr}\right) + \frac{1}{r^2}\,G(r)\,\frac{d^2\psi}{d\theta^2} = 0 \quad \Big| \; \frac{1}{\frac{1}{r^2}\,G(r)\,\psi(\theta)}$$

$$\frac{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right)}{\frac{1}{r^2}\,G(r)} = -\frac{d^2\psi/d\theta^2}{\psi(\theta)} = \lambda$$

$$u(r,0) = 0,\quad u\left(r,\frac{\pi}{3}\right) = 0 \;\Rightarrow\; \psi(0) = 0,\; \psi\left(\frac{\pi}{3}\right) = 0$$

$$\begin{cases} \dfrac{d^2\psi}{d\theta^2} + \lambda\,\psi(\theta) = 0 \\ \psi(0) = 0, \quad \psi\left(\frac{\pi}{3}\right) = 0 \end{cases}$$

$$r\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) - \lambda\,G(r) = 0$$

<u>ψ-problem</u> : oscillatory in θ $\Rightarrow$ $\lambda > 0$

$\psi(\theta) = C_1 \cos(\sqrt{\lambda}\,\theta) + C_2 \sin(\sqrt{\lambda}\,\theta)$

$\psi(0) = 0 \Rightarrow C_1 = 0 \Rightarrow \psi(\theta) = \underset{\neq 0}{C_2} \sin(\sqrt{\lambda}\,\theta)$

for a trivial solutions

$\psi(\frac{\pi}{3}) = 0 \Rightarrow \sin(\sqrt{\lambda}\,\frac{\pi}{3}) = 0$

$\therefore \sqrt{\lambda}\,\frac{\cancel{\pi}}{3} = \cancel{\pi} n, \quad n = 1, 2, \ldots$

$\sqrt{\lambda} = 3n \Rightarrow \lambda_n = 9n^2, \quad n = 1, 2, \ldots$: <u>e'values</u>

$\psi_n(\theta) = \sin(3n\theta)$: <u>e'functions</u>

<u>G-problem</u>

$r\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) - \lambda G(r) = 0$

$r\left(\frac{dG}{dr} + r\frac{d^2G}{dr^2}\right) - \lambda G(r) = 0$

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$$r^2 \frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda G(r) = 0$$ equipotential or Euler eq^n

power of r = order of derivative

let $G(r) = r^p$ $\quad G' = p r^{p-1}$ $\quad G'' = p(p-1) r^{p-2}$

$$r^2 p(p-1) r^{p-2} + r p r^{p-1} - \lambda r^p = 0$$

$$\underbrace{r^p}_{\neq 0} \left[p(p-1) + p - \lambda \right] = 0$$

$$p^2 - \cancel{p} + \cancel{p} - \lambda = 0$$: characteristic eq^n

$$p^2 = \lambda > 0 \Rightarrow p = \pm\sqrt{\lambda} = \pm 3n \Rightarrow r^{3n} \ \& \ r^{-3n}$$

$$G_n(r) = a_n r^{3n} + b_n r^{-3n}$$

solution $u(r,\theta)$ should be bounded in the domain (wedge) including at the origin $r=0 \Rightarrow b_n = 0$

$r=0$

Using principle of linear superposition:

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} B_n \cdot r^{3n} \sin(3n\theta)}$$

We compute coefficients B_n using BC at $r=a$

$u(a,\theta) = f(\theta)$

and orthogonality of sines:

$$u(a,\theta) = \sum_{n=1}^{\infty} \underbrace{B_n \cdot a^{3n}}_{\substack{\text{coefficients of} \\ \text{Fourier sine series}}} \cdot \sin(3n\theta) = f(\theta) \qquad / \cdot \sin(3m\theta)$$

$$\int_0^{\pi/3} \sin(3n\theta) \cdot \sin(3m\theta)\, d\theta = \begin{cases} 0, & n \neq m \\ \frac{\pi}{6}, & n = m \neq 0 \end{cases}$$

$$B_n a^{3n} \cdot \frac{\pi}{6} = \int_0^{\pi/3} f(\theta) \sin(3n\theta)\, d\theta$$

$$B_n = \dots$$

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#1
HW #7

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots \equiv S = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Fourier cosine series for x:

$$x = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi x}{L} + \frac{\cos 3\pi x/L}{3^2} + \frac{\cos 5\pi x/L}{5^2} + \ldots\right) \quad 0 \le x \le L$$

at $x=0$:

$$0 = \frac{L}{2} - \frac{4L}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots = \frac{\pi^2}{8}$$

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots = \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{1}{4}\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{S} = \frac{1}{4}S$$

since $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots = S$ and

$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots = \frac{1}{4}S \Rightarrow$ the rest is

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots = \frac{3}{4}S$$

But $1+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8} \Rightarrow \frac{3}{4}S=\frac{\pi^2}{8} \Rightarrow \boxed{S=\frac{\pi^2}{6}}$

The midterm exam will include problems on

- separation of variables
- term-by-term differentiation / integration of a Fourier series
- convergence of Fourier series
- other problems
- maybe a bonus problem similar to problem #1 from HW #7
- odd and even parts, odd & even extensions
- conservation of energy, modes for wave eq^{2}
- standing & traveling waves

#2 HW #6

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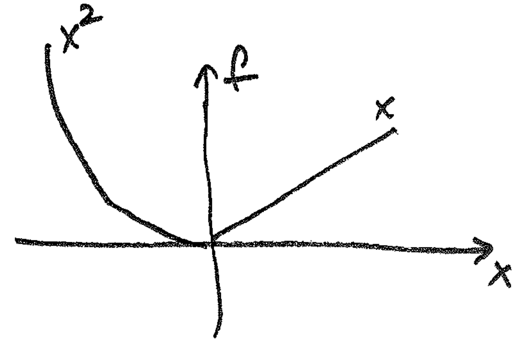
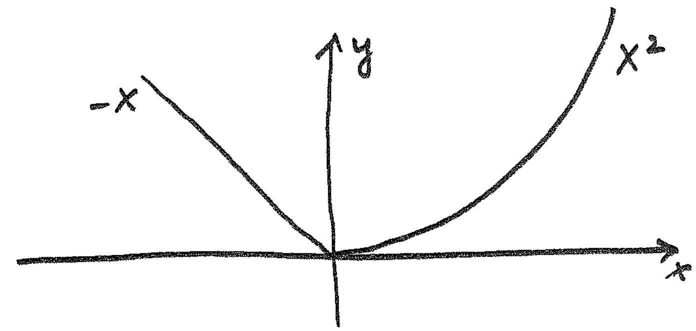
Even and Odd Extensions. Even and odd parts.

$$f(x) = \begin{cases} x, & x>0 \\ x^2, & x<0 \end{cases}$$



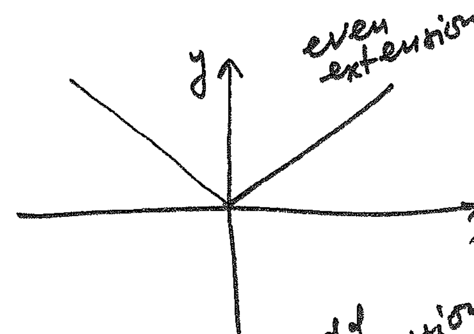
f(−x)

$$f(-x) = \begin{cases} -x, & -x>0 \quad \text{or} \quad x<0 \\ (-x)^2 = x^2, & -x<0 \quad \text{or} \quad x>0 \end{cases}$$



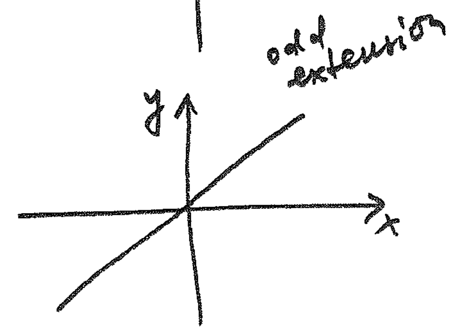
Even extension of f is

$$\begin{cases} f(x), & x>0 \\ f(-x), & x<0 \end{cases} = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$$



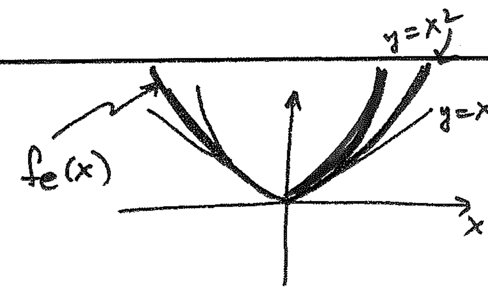
Odd extension of f is

$$\begin{cases} f(x), & x>0 \\ -f(-x), & x<0 \end{cases} = \begin{cases} x, & x>0 \\ -(-x) = x, & x<0 \end{cases}$$



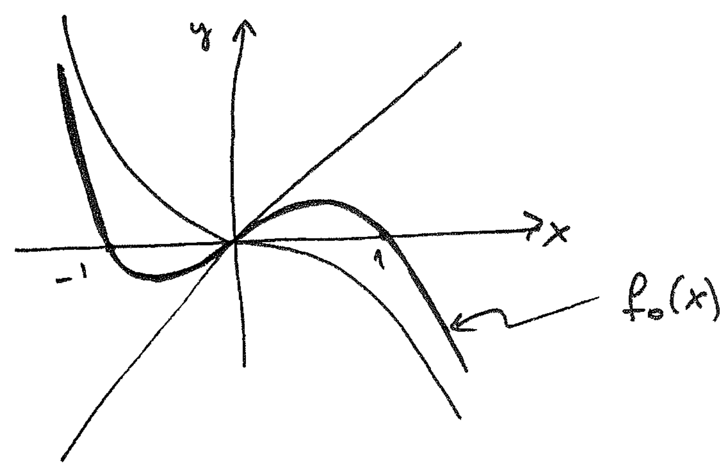
$$f_e(x) = \frac{1}{2}\left[f(x) + f(-x) \right] = \frac{1}{2}\begin{cases} x + x^2, & x > 0 \\ x^2 - x, & x < 0 \end{cases}$$

even part of f



$$f_o(x) = \frac{1}{2}\left[f(x) - f(-x) \right] = \frac{1}{2}\begin{cases} x - x^2, & x > 0 \\ x^2 - (-x) = x^2 + x, & x < 0 \end{cases}$$

odd part of f



Back to Sturm-Liouville theory (Ch 5)

* Circularly symmetric heat flow

(example when material properties are constant but equation has variable coefficients)

$$u_t = K \nabla^2 u \qquad u = u(r, \theta, t)$$

Recall $\nabla^2 u = \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$

Circularly symmetric heat flow or radially symmetric:

$\Rightarrow u_\theta = 0 \rightarrow u = u(r,t) \Rightarrow u_t = \frac{K}{r}(r u_r)_r$

Separation of variables: $u(r,t) = \phi(r)\, h(t)$

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$$\phi(r)\frac{dh}{dt} = \frac{K}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)h(t) \quad \Big| \quad \frac{1}{\phi h K}$$

$$\frac{dh/dt}{Kh} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\frac{1}{\phi} = -\lambda$$

$$\boxed{\begin{aligned} &\frac{dh}{dt} + \lambda K h = 0 \\ &\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda\phi = 0 \end{aligned}}$$

$$\therefore \quad h(t) = Ce^{-\lambda K t}$$

$$\phi(r) - ?$$

Sturm-Liouville eigenvalue problem

Consider a differential equation (DE)

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx} \right) + q(x)\,\phi(x) + \lambda\,\sigma(x)\,\phi(x) = 0, \quad a < x < b$$

This DE is linear, homogeneous, 2^{nd} order with variable coefficients.

λ: eigenvalue

$\phi(x)$: associated eigenfunction, $\phi(x) \not\equiv 0$

$\sigma(x)$: weight function

Examples

1. $\phi'' + \lambda\phi = 0$ $\qquad p=1 \qquad q=0 \qquad \sigma=1$

2. Heat flow in a non-uniform rod:

$$\frac{d}{dx}\left(K_0(x)\frac{d\phi}{dx}\right) + \alpha(x)\phi(x) + \lambda c(x)\rho(x)\phi(x) = 0$$

$$p(x) = K_0(x) \qquad q(x) = \alpha(x) \qquad \sigma(x) = c(x)\rho(x)$$

3. Circularly symmetric ~~heat~~ flow:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda\phi = 0 \quad / \cdot r$$

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0$$

$$p(r) = r, \qquad q(r) = 0 \qquad \sigma(r) = r$$