Lecture 26

Midterm Review

Exam covers Chapters 1-3.

Ch 1 Heat equation

- no derivation
- steady state solution
- various boundary conditions, physical meaning
- properties of solutions (oscillatory, exp decay)

Ch 2 Separation of variables

- in Cartesian coordinates (x,y) (or in (t,x))
- in polar coordinates (r,θ)
- principle of linear superposition
- orthogonality of eigenfunctions

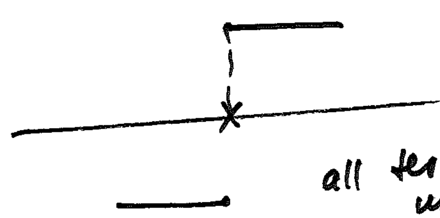
- review standard problems on separation of variables that lead to Fourier sine, Fourier cosine and Fourier series

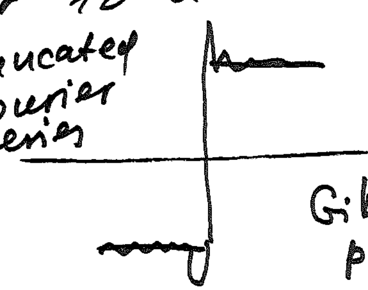
- no fluid problem

- Laplace equation, properties of solutions (oscillatory, non-oscillatory etc.)

Ch3 Fourier series

- Fourier sine, Fourier cosine, Fourier series
- convergence of Fourier sine, cosine and Fourier series
- periodic extensions, even and odd periodic extensions
- graphs of Fourier series including graphs of truncated Fourier series, say, the first 10 or 100 terms



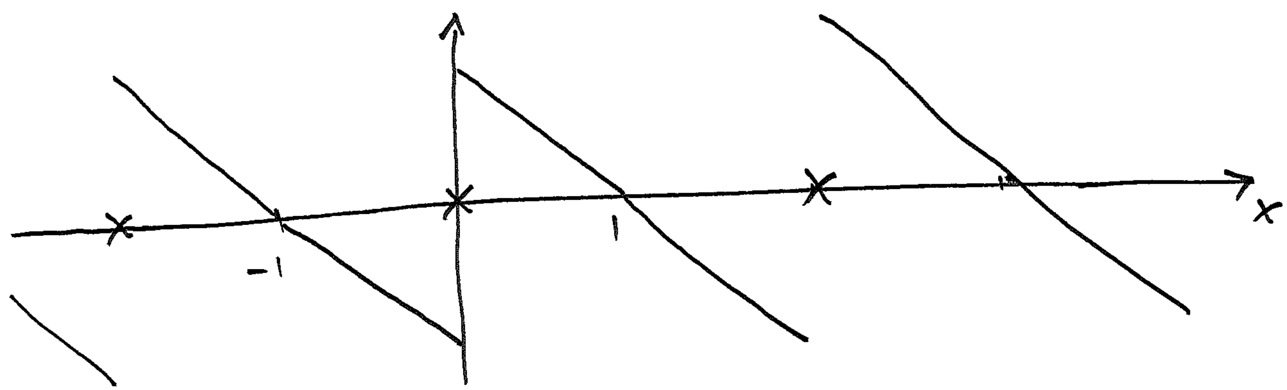


- odd and even parts of a function
- term-by-term differentiation and integration of a Fourier series
- complex form of Fourier series

You may bring one page, one sided, letter size, of your notes

<u>Ex</u> $f(x) = 1-x, \quad 0 \le x \le 1$

Sketch Fourier sine series



graph of a Fourier sine series

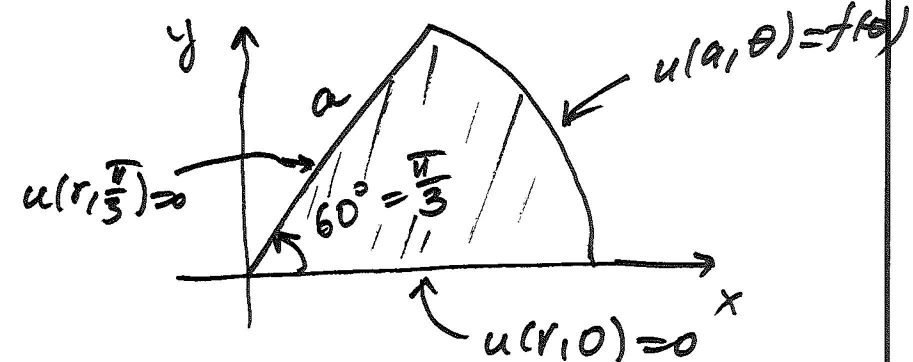
$$f_o(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

$$f(x) = 1 - x \qquad f(-x) = 1 + x$$

$$f_o(x) = \frac{1}{2}\left(1 - x - (1+x)\right) = \frac{1}{2}(-2x) = -x$$

Fourier series of $f_o(x) = -x$ is Fourier sine series since $-x$ is odd and all coefficients of cosine terms $A_n = 0$.

__Ex__ Using separation of variables, solve Laplace's equation inside a 60° wedge of radius a subject to the following boundary conditions:



$$u(r, 0) = 0$$

$$u(r, \tfrac{\pi}{3}) = 0$$

$$u(a, \theta) = f(\theta)$$

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$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

separation of variables: $u(r,\theta) = G(r)\,\Psi(\theta)$

$$\nabla^2 u = 0$$

$$\Psi(\theta)\,\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) + \frac{1}{r^2}\,G(r)\,\frac{d^2\Psi}{d\theta^2} = 0 \quad \Bigg| \; \frac{1}{\frac{1}{r^2}G(r)\Psi(\theta)}$$

$$\frac{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right)}{\frac{1}{r^2}G(r)} = -\frac{d^2\Psi/d\theta^2}{\Psi(\theta)} = \lambda$$

$$u(r,0)=0, \quad u\left(r,\frac{\pi}{3}\right)=0 \Rightarrow \Psi(0)=0, \quad \Psi\left(\frac{\pi}{3}\right)=0$$

$$\begin{cases} \dfrac{d^2\Psi}{d\theta^2} + \lambda\,\Psi(\theta) = 0 \\ \Psi(0)=0, \quad \Psi\left(\frac{\pi}{3}\right)=0 \end{cases}$$

$$r\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) - \lambda\,G(r) = 0$$

Ψ-problem: oscillatory in θ $\Rightarrow \lambda > 0$

$$\Psi(\theta) = C_1 \cos(\sqrt{\lambda}\,\theta) + C_2 \sin(\sqrt{\lambda}\,\theta)$$

$$\Psi(0) = 0 \Rightarrow C_1 = 0 \Rightarrow \Psi(\theta) = \underset{\neq 0}{C_2} \sin(\sqrt{\lambda}\,\theta)$$

for a trivial solution

$$\Psi\left(\frac{\pi}{3}\right) = 0 \Rightarrow \sin\left(\sqrt{\lambda}\,\frac{\pi}{3}\right) = 0$$

$$\therefore \sqrt{\lambda}\,\frac{\cancel{\pi}}{3} = \cancel{\pi} n, \quad n = 1, 2, \ldots$$

$$\sqrt{\lambda} = 3n \Rightarrow \lambda_n = 9n^2, \quad n = 1, 2, \ldots$$: e'values

$$\Psi_n(\theta) = \sin(3n\theta)$$: e'functions

G-problem

$$r\frac{d}{dr}\left(r\frac{d}{dr}G(r)\right) - \lambda G(r) = 0$$

$$r\left(\frac{dG}{dr} + r\frac{d^2G}{dr^2}\right) - \lambda G(r) = 0$$

$$r^2 \frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda G(r) = 0$$ equipotential or Euler eq$^{\underline{n}}$

power of r = order of derivative

let $G(r) = r^p$ $\quad G' = p r^{p-1}$ $\quad G'' = p(p-1)\, r^{p-2}$

$$r^2 p(p-1)\, r^{p-2} + r p r^{p-1} - \lambda r^p = 0$$

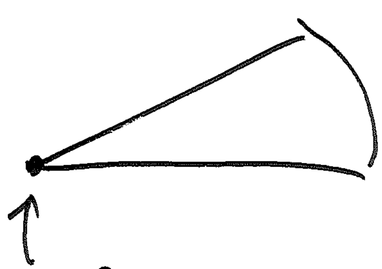
$$\underbrace{r^p}_{\neq 0}\left[p(p-1) + p - \lambda \right] = 0$$

$$p^2 - \cancel{p} + \cancel{p} - \lambda = 0$$: characteristic eq$^{\underline{n}}$

$$p^2 = \lambda > 0 \Rightarrow p = \pm\sqrt{\lambda} = \pm 3n \Rightarrow r^{3n} \ \& \ r^{-3n}$$

$$G_n(r) = a_n r^{3n} + b_n r^{-3n}$$

solution $u(r,\theta)$ should be bounded in the domain (wedge) including at the origin $r=0 \Rightarrow b_n = 0$



$r=0$

Using principle of linear superposition:

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} B_n \cdot r^{3n} \sin(3n\theta)}$$

We compute coefficients B_n using BC at $r=a$

$u(a,\theta) = f(\theta)$

and orthogonality of sines:

$$u(a,\theta) = \sum_{n=1}^{\infty} \underbrace{B_n \cdot a^{3n}}_{\substack{\text{coefficients of} \\ \text{Fourier sine series}}} \cdot \sin(3n\theta) = f(\theta) \quad \Big/ \cdot \sin(3m\theta)$$

$$\int_0^{\pi/3} \sin(3n\theta) \cdot \sin(3m\theta)\, d\theta = \begin{cases} 0, & n \neq m \\ \frac{\pi}{6}, & n = m \neq 0 \end{cases}$$

$$B_n a^{3n} \cdot \frac{\pi}{6} = \int_0^{\pi/3} f(\theta) \sin(3n\theta)\, d\theta$$

$$B_n = \ldots$$

#4
HW #7

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots \equiv S = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Fourier cosine series for x:

$$x = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi x}{L} + \frac{\cos 3\pi x/L}{3^2} + \frac{\cos 5\pi x/L}{5^2} + \dots\right) \quad 0 \le x \le L$$

at $x=0$:

$$0 = \frac{L}{2} - \frac{4L}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\dots = \frac{\pi^2}{8}$$

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\dots = \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{1}{4}\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{S} = \frac{1}{4}S$$

since $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots = S$ and

$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\dots = \frac{1}{4}S \Rightarrow$ the rest is

$1+\frac{1}{3^2}+\frac{1}{5^2}+\dots = \frac{3}{4}S$

But $1+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8} \Rightarrow \frac{3}{4}S=\frac{\pi^2}{8} \Rightarrow \boxed{S=\frac{\pi^2}{6}}$

The midterm exam will include problems on

- separation of variables
- term-by-term differentiation / integration of a Fourier series
- convergence of Fourier series
- other problems
- maybe a bonus problem similar to problem #4 from HW #7