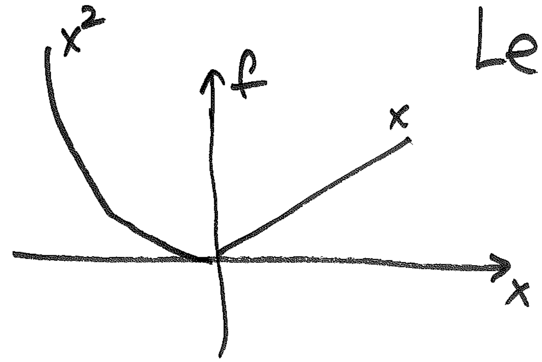
University of Idaho

Lecture 27

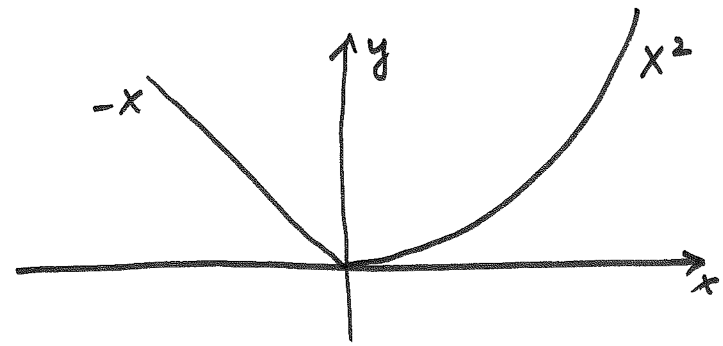
Even and Odd Extensions. Even and odd parts.

$$f(x) = \begin{cases} x, & x>0 \\ x^2, & x<0 \end{cases}$$



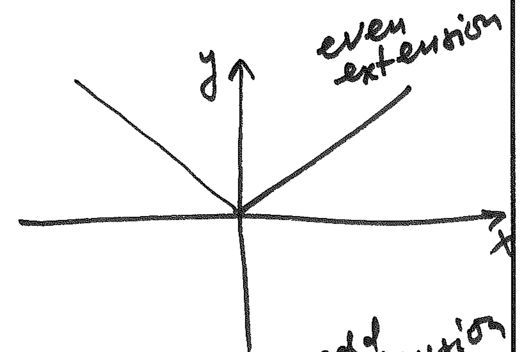
f(−x)

$$f(-x) = \begin{cases} -x, & -x>0 \quad \text{or} \quad x<0 \\ (-x)^2 = x^2, & -x<0 \quad \text{or} \quad x>0 \end{cases}$$



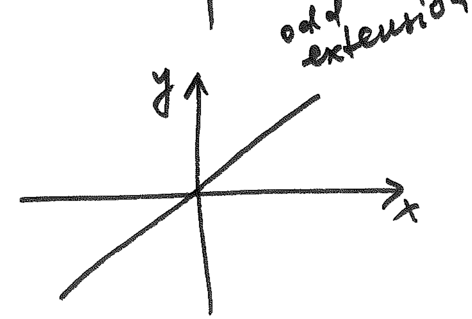
Even extension of f is

$$\begin{cases} f(x), & x>0 \\ f(-x), & x<0 \end{cases} = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$$



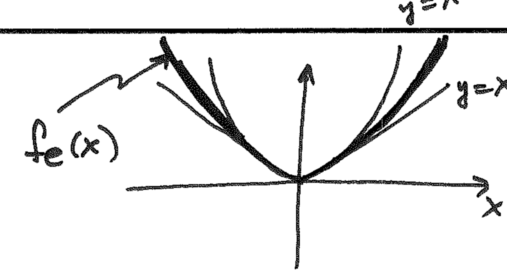
Odd extension of f is

$$\begin{cases} f(x), & x>0 \\ -f(-x), & x<0 \end{cases} = \begin{cases} x, & x>0 \\ -(-x) = x, & x<0 \end{cases}$$



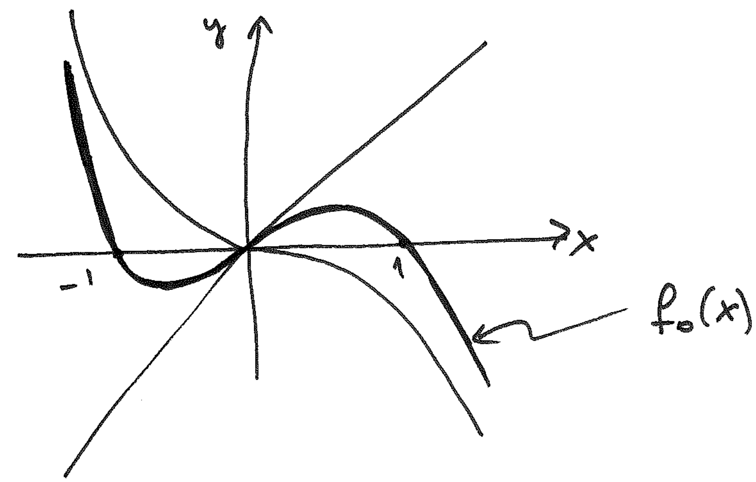
$$f_e(x) = \frac{1}{2}\left[f(x) + f(-x) \right] = \frac{1}{2}\begin{cases} x + x^2, & x > 0 \\ x^2 - x, & x < 0 \end{cases}$$

even part of f



$$f_o(x) = \frac{1}{2}\left[f(x) - f(-x) \right] = \frac{1}{2}\begin{cases} x - x^2, & x > 0 \\ x^2 - (-x) = x^2 + x, & x < 0 \end{cases}$$

odd part of f



Back to Sturm-Liouville theory

* <u>Heat flow in a non-uniform rod</u>

$$c\rho u_t = (K_0 u_x)_x + Q$$

where $c = c(x)$, $\rho = \rho(x)$, $K_0 = K_0(x)$

Let $Q = \alpha(x)\, u(x,t)$: heat source depends on temperature

Then

$c\rho u_t = (K_0 u_x)_x + \alpha(x) u$: linear homogeneous eq^{n}

Separation of variables:

$$u(x,t) = \phi(x)\, h(t)$$

$$c(x)\rho(x)\phi(x)\frac{dh}{dt} = h(t)\frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) + \alpha(x)\phi(x)h(t) \Big/ \cdot \frac{1}{c\rho\phi h}$$

$$\frac{1}{h}\frac{dh}{dt} = \frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right)\frac{1}{c(x)\rho(x)\phi(x)} + \alpha(x)\frac{1}{c(x)\rho(x)} = -\lambda$$

$$\boxed{\begin{gathered}\frac{dh}{dt} + \lambda h = 0 \\ \frac{d}{dx}\left(K_0(x)\frac{d\phi}{dx}\right) + \alpha(x)\phi(x) + \lambda c(x)\rho(x)\phi(x) = 0\end{gathered}}$$

$$h(t) = Ce^{-\lambda t}$$

$\phi(x)$ - ? Cannot solve in general

Goal: understand behaviour of $\phi(x)$ without having it in a closed form

* Circularly symmetric heat flow (example when physical properties are constant but equation has variable coefficients)

Recall

$$\nabla^2 u = \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$$

Circularly symmetric or radially symmetric problem:

$$u = u(r,t)$$

$$\therefore \quad u_t = \frac{K}{r}(r u_r)_r \qquad \text{where} \quad K = \frac{K_0}{c\rho} : \text{const}$$

Separation of variables:

$$u(r,t) = \phi(r)\, h(t)$$

$$\phi(r)\frac{dh}{dt} = \frac{K}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) h(t) \qquad \Big/ \ \frac{1}{\phi h K}$$

$$\frac{dh/dt}{Kh} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\frac{1}{\phi} = -\lambda$$

$$\boxed{\begin{aligned} &\frac{dh}{dt} + \lambda K h = 0 \\ &\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda\phi = 0 \end{aligned}}$$

University of Idaho

$\therefore \quad h(t) = C e^{-\lambda k t}$

$\phi(r)$ - ? cannot be solved in general

Sturm-Liouville eigenvalue problem

Consider a differential equation (DE)

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\,\phi(x) + \lambda\,\sigma(x)\,\phi(x) = 0$$

This DE is linear, homogeneous but has variable coefficients.

λ: eigenvalue

$\phi(x)$: associated eigenfunction, $\phi(x) \not\equiv 0$

Equation is valid on some interval $a < x < b$

Examples

1. $\phi'' + \lambda\phi = 0$ $\qquad p = 1, \quad q = 0, \quad \sigma = 1$

2. Heat flow in a non-uniform rod:

$$\frac{d}{dx}\left(K_0(x)\frac{d\phi}{dx}\right) + \alpha(x)\phi(x) + \lambda c(x)\rho(x)\,\phi(x) = 0$$

$p(x) = K_0(x), \quad q(x) = \alpha(x), \quad \sigma(x) = c(x)\rho(x)$

3. Circularly symmetric heat flow:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda\phi = 0 \qquad / \cdot r$$

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0$$

$p(r) = r, \quad q(r) = 0, \quad \sigma(r) = r$

Boundary conditions

	Heat eq^{n}	Wave eq^{n}	Math name
$\phi = 0$	Fixed temperature	Fixed displacement	Dirichlet BC or BC of the 1st kind
$\frac{d\phi}{dx} = 0$	Insulated	Free end	Neumann BC or BC of the 2nd kind
$\frac{d\phi}{dx} = \pm h\phi$	Newton's law of Cooling	Elastic	Mixed BC or Robin BC or BC of the 3rd kind
$\phi(-L) = \phi(L)$ and $\frac{d\phi(-L)}{dx} = \frac{d\phi(L)}{dx}$	periodic	periodic	periodic
$\lvert\phi(0)\rvert < \infty$	bounded at the origin	bounded at the origin	singularity condition

Thm Regular Sturm-Liouville problem

1. All eigenvalues λ are real
2. There are infinitely many eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \lambda_{n+1} < \dots$$

(a) There is the smallest e'value λ_1

(b) There is no largest e'value, i.e.

$$\lim_{n\to\infty} \lambda_n = +\infty$$

3. To every e'value λ_n, there exists an associated eigenfunction $\phi_n(x)$, defined up to a multiplicative constant $(\neq 0)$

$\phi_n(x)$ has exactly $n-1$ roots in $a < x < b$

4. The eigenfunctions $\phi_n(x)$ form a <u>complete</u> set in a sense that any piecewise smooth function $f(x)$ can be represented a generalized Fourier series

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$ <u>generalized Fourier series</u>

Furthermore, this series converges to

$\frac{1}{2}[f(x^+) + f(x^-)]$ for $a < x < b$