Lecture 27

Boundary conditions

	Heat eq^{n}	Wave eq^{n}	Math name
$\phi = 0$	Fixed temperature	Fixed displacement	Dirichlet BC or BC of the 1st kind
$\frac{d\phi}{dx} = 0$	Insulated	Free end	Neumann BC or BC of the 2nd kind
$\frac{d\phi}{dx} = \pm h\phi$	Newton's law of Cooling	Elastic.	Mixed BC or Robin BC or BC of the 3rd kind
$\phi(-L) = \phi(L)$ and $\frac{d\phi(-L)}{dx} = \frac{d\phi(L)}{dx}$	periodic	periodic	periodic
$\lvert\phi(0)\rvert < \infty$	bounded at the origin	bounded at the origin	singularity condition

Thm Regular Sturm - Liouville problem

Def Regular Sturm–Liouville problem is

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0, \qquad a < x < b$$

$$\left.\begin{aligned} \beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) &= 0 \\ \beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) &= 0 \end{aligned}\right\}$$

General BCs:

1^{st}, 2^{nd} and 3^{rd} kinds of BCs.

$\beta_1, .., \beta_4$: real numbers

Functions $p(x)$, $q(x)$, $\sigma(x)$ are real-valued and continuous on $a \leq x \leq b$. In addition, $p(x) > 0$, $\sigma(x) > 0$ and $a \leq x \leq b$.

<u>Thm</u> <u>Regular Sturm-Liouville problem</u>

1. All eigenvalues λ are real.
2. There are infinitely many eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \lambda_{n+1} < \dots$$

(a) There is the smallest e'value λ_1

(b) There is no largest e'value, ie

$$\lim_{n \to \infty} \lambda_n = +\infty$$

3. To every e'value λ_n, there exists an associated e'function $\phi_n(x)$, defined up to a multiplicative constant $(\neq 0)$.

$\phi_n(x)$ has exactly $n-1$ roots in $a < x < b$

4. The e'functions $\phi_n(x)$ form a <u>complete set</u> in a sense that any piecewise smooth function $f(x)$ can be represented by a <u>generalized</u> Fourier series

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$

Furthermore, this series converges to

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right] \quad \text{for} \quad a < x < b$$

5. The eigenfunctions belonging to different e'values are orthogonal relative to the weight function $\sigma(x)$:

$$\int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx = 0 \quad \text{if } n \neq m$$

$\langle \phi_n, \phi_m \rangle$: inner product of ϕ_n and ϕ_m

6. Any eigenvalue can be related to eigenfunction through Rayleigh Quotient:

$$\lambda = \frac{-p\phi\phi'\Big|_a^b + \int_a^b \left[p(\phi')^2 - q\phi\right]dx}{\int_a^b \phi^2 \sigma\, dx}$$

Consider a special case:

Sturm-Liouville problem: $\phi'' + \lambda\phi = 0$

$\phi(0)=0, \quad \phi(L)=0$

$p(x)=1, \quad q(x)=0, \quad \sigma(x)=1, \quad a=0, \quad b=L$

We showed previously $\lambda_n = \left(\frac{n\pi}{L}\right)^2 > 0$, $n=1,2,\ldots$ and they are real

1. Real e'values: we showed that e'values $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ are real and positive, otherwise there was no nontrivial function $\phi(x)$.

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x) \quad / \ \phi_m(x)\,\sigma(x) \quad \int_a^b$$

$$\int_a^b f(x)\,\phi_m(x)\,\sigma(x)\,dx = \int_a^b \sum_{n=1}^{\infty} a_n \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx$$

$$= \sum_{n=1}^{\infty} a_n \underbrace{\int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx}_{=0 \text{ if } n \neq m} =$$

$$= a_m \int_a^b \phi_m^2(x)\,\sigma(x)\,dx$$

$$\therefore \quad \boxed{a_m = \frac{\int_a^b f(x)\,\phi_m(x)\,\sigma(x)\,dx}{\int_a^b \phi_m^2(x)\,\sigma(x)\,dx}}$$

We assumed that λ_n were real but now with Sturm-Liouville Thm we know that λ's are real (Thm proves that).

2. <u>Ordering e'values</u>:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$

$$\lambda_1 = \left(\frac{\pi}{L}\right)^2 : \text{ the smallest e'value}$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \longrightarrow \infty \quad \text{as} \quad n \to \infty$$

3. <u>e' functions:</u> $\phi_n(x) = \sin \frac{n\pi x}{L}$, $0 \le x \le L$

In the interval $0 < x < L$, $\sin \frac{n\pi x}{L}$ has exactly $n-1$ roots.

4. <u>Complete set:</u> any piecewise smooth function $f(x)$ can be represented as a Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \underbrace{\sin \frac{n\pi x}{L}}_{\phi_n(x)} = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

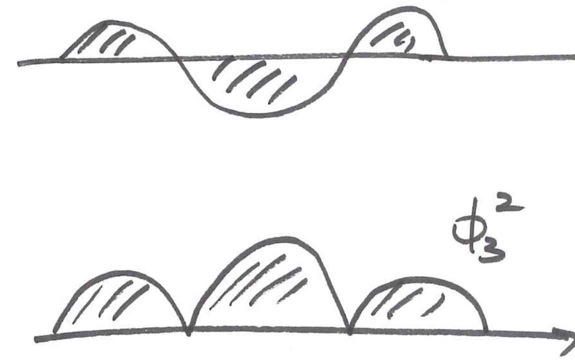
5. <u>Orthogonality of e'functions</u>

In general,

$$\int_a^b \phi_n(x)\, \phi_m(x)\, \sigma(x)\, dx = 0 \qquad \text{if } n \ne m$$

$m \to n$

$$\boxed{a_n = \frac{\int_a^b f(x)\,\phi_n(x)\,\sigma(x)\,dx}{\int_a^b \underbrace{\phi_n^2(x)}_{\geq 0}\,\underbrace{\sigma(x)}_{>0}\,dx}} \quad > 0$$



Our special case: $\phi_n(x) = \sin\frac{n\pi x}{L}$

$\sigma(x) = 1$

$a = 0, \quad b = L$

$$a_n = \frac{\int_0^L f(x)\sin\frac{n\pi x}{L}\cdot 1\,dx}{\int_0^L \sin^2\frac{n\pi x}{L}\cdot 1\,dx}$$

$$\int_0^L \sin^2 \frac{n\pi x}{L}\,dx = \frac{L}{2}$$

$$\therefore \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad : \quad \text{same as before}$$

6. <u>Rayleigh Quotient</u> We will prove later that e'value λ and associated e'function $\phi(x)$ are related by

$$\lambda = \frac{-p\,\phi\,\phi'\Big|_a^b + \int_a^b \left[p(\phi')^2 - q\,\phi^2\right]dx}{\int_a^b \phi^2\,\sigma\,dx}$$

<u>Special case</u>: $\phi(0) = \phi(L) = 0$ $\quad p = 1,\ q = 0,\ \sigma = 1$

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$$\therefore \quad \lambda = \frac{0 + \int_0^L (\phi')^2\,dx}{\underbrace{\int_0^L \phi^2\,dx}_{>0}} = \frac{\int_0^L (\phi')^2\,dx}{\int_0^L \phi^2\,dx}$$

$$(\phi')^2 \geq 0 \Rightarrow \int_0^L (\phi')^2\,dx \geq 0 \Rightarrow \boxed{\lambda \geq 0}$$

Q can λ be 0 ?

let $\lambda = 0$ be an e'value

$$0 = \frac{\int_0^L (\phi')^2\,dx}{\underbrace{\int_0^L \phi^2\,dx}_{>0}} \Rightarrow \int_0^L \underbrace{(\phi')^2}_{\geq 0}\,dx \Rightarrow \phi' \equiv 0$$

$\rightarrow \phi = \text{const}$ but $\phi(0) = \phi(L) = 0 \Rightarrow \phi \equiv 0$

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$\therefore\ \lambda = 0$ is not an e'value.

Hence,

$$\lambda > 0$$

<u>Note</u>: we showed that e'values $\lambda > 0$ without knowing $\phi(x)$ or λ explicitly!