Lecture 28

Def <u>Regular Sturm-Liouville Problem</u> is

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0, \qquad a < x < b$$

$$\left.\begin{aligned} \beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) &= 0 \\ \beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) &= 0 \end{aligned}\right\} \quad \text{General BCs: } 1^{st}, 2^{nd}, 3^{rd} \text{ kind BCs}$$

β_i: real numbers

Functions $p(x)$, $q(x)$, $\sigma(x)$ are real-valued and continuous on $a \le x \le b$. In addition, $p(x) > 0$, $\sigma(x) > 0$ on $a \le x \le b$

<u>Thm</u> <u>Regular Sturm - Liouville problem</u> (Cont'd)

5. The eigenfunctions belonging to different e'values are orthogonal relative to the weight function $\sigma(x)$:

$$\int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx = 0 \quad \text{if } n \neq m$$

6. Any eigenvalue can be related to eigenfunction through the <u>Raileigh Quotient</u>:

$$\lambda = \frac{-p\,\phi\,\phi'\Big|_a^b + \int_a^b \left[p\,(\phi')^2 - q\,\phi \right] dx}{\int_a^b \phi^2\,\sigma\,dx}$$

Consider a special case:

Sturm - Liouville problem: $\quad \phi''(x) + \lambda\,\phi(x) = 0$

$\phi(0) = 0, \quad \phi(L) = 0$

$p(x) = 1, \quad q(x) = 0, \quad \sigma(x) = 1, \quad a = 0, \quad b = L$

1. **Real e'values**: we showed $\lambda > 0$, otherwise there is only a trivial solution $\phi(x)$.

We assumed that λ was real, but now Thm on Regular Sturm-Liouville problem proves that λ is real.

2. **Ordering e'values:**

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$

$\lambda_1 = \left(\frac{\pi}{L}\right)^2$: the smallest e'value

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \to \infty \quad \text{as} \quad n \to \infty$$

3. **e'functions**: $\phi_n(x) = \sin \frac{n\pi x}{L}$ $\qquad 0 \le x \le L$

In the interval $0 < x < L$, there are exactly $n-1$ roots

4. <u>Complete set</u> : any piecewise smooth function can be represented as a Fourier sine series

$$f(x) \sim \sum_{n=1}^{\infty} a_n \underbrace{\sin \frac{n\pi x}{L}}_{\phi_n(x)} = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

5. <u>Orthogonality of eigenfunctions</u>

In general,

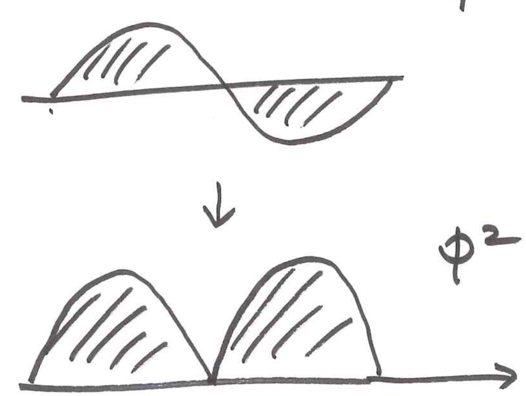
$$\int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x) = 0 \qquad \text{if } n \neq m$$

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad / \ \phi_m(x)\,\sigma(x)$$

$$\int_a^b f(x)\,\phi_m(x)\,\sigma(x)\,dx = \int_a^b \sum_{n=1}^{\infty} a_n \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx =$$

$$= \sum_{n=1}^{\infty} a_n \underbrace{\int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx}_{=\,0 \text{ if } n \neq m} = a_m \int_a^b \phi_m^2(x)\,\sigma(x)\,dx$$

$$\therefore \boxed{a_m = \frac{\int_a^b f(x)\,\phi_m(x)\,\sigma(x)\,dx}{\int_a^b \underbrace{\phi_m^2(x)}_{\geq 0}\,\underbrace{\sigma(x)}_{>0}\,dx}} \quad >0$$



Special case: $\phi_m(x) = \sin\frac{m\pi x}{L}$

$\sigma(x) = 1$

$a = 0, \quad b = L$

$$a_m = \frac{\int_0^L f(x)\sin\frac{m\pi x}{L}\cdot 1\,dx}{\int_0^L \sin^2\frac{m\pi x}{L}\cdot 1\,dx}$$

$$\int_0^L \sin^2 \frac{m\pi x}{L}\, dx = \frac{L}{2}$$

$$\therefore \quad a_m = \frac{2}{L}\int_0^L f(x) \sin \frac{m\pi x}{L}\, dx$$: same as before

6. <u>Rayleigh Quotient</u> We will prove later that e'value λ and associated e'function $\phi(x)$ are related by

$$\lambda = \frac{-p\phi\phi'\Big|_a^b + \int_a^b \left[p(\phi')^2 - q\phi^2\right] dx}{\int_a^b \phi^2 \sigma\, dx}$$

<u>Special case</u>: $\phi(0) = \phi(L) = 0$, $p = 1$, $q = 0$, $\sigma = 1$

$$\therefore \quad \lambda = \frac{0 + \int_0^L (\phi')^2 dx}{\underbrace{\int_0^L \phi^2 dx}_{>0}} = \frac{\int_0^L (\phi')^2 dx}{\int_0^L \phi^2 dx}$$

$(\phi')^2 \geq 0 \quad \Rightarrow \quad \lambda \geq 0$

Let $\lambda = 0 \Rightarrow (\phi')^2 \equiv 0 \Rightarrow \phi' \equiv 0 \Rightarrow \phi(x) = \text{const}$

but $\phi(0) = \phi(L) = 0 \Rightarrow \text{const} = 0 \Rightarrow \phi(x) \equiv 0$ trivial solution ↯

$\therefore \lambda = 0$ is not an e'value

$\Rightarrow \lambda > 0$

Note: we showed that $\lambda > 0$ without knowing $\phi(x)$ explicitly!

Ex Heat flow in a nonuniform rod

$c\rho u_t = (K_0 u_x)_x \quad$ in $\quad 0 < x < L, \quad t > 0$

$u(0,t) = 0$

$u_x(L,t) = 0$

$u(x,0) = f(x)$

Separation of variables: $u(x,t) = \phi(x)\, h(t)$

$$c\rho\, \phi(x) \frac{dh}{dt} = h(t) \cdot \frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) \quad \Big| \quad \frac{1}{c\rho\, \phi\, h}$$

$$\frac{dh/dt}{h} = \frac{\frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right)}{c(x)\rho(x)\, \phi(x)} = -\lambda$$

$$\therefore \quad \frac{dh}{dt} + \lambda h = 0 \quad \Rightarrow \quad h(t) = C e^{-\lambda t}$$

$$\begin{cases} \frac{d}{dx}\left(K_0(x) \frac{d\phi}{dx}\right) + \lambda c(x)\rho(x)\phi(x) = 0 \\ \phi(0) = 0, \quad \frac{d\phi}{dx}(L) = 0 \end{cases}$$

$p(x) = K_0(x) > 0$
$q(x) = 0$
$\sigma(x) = c(x)\rho(x) > 0$

This is a regular Sturm-Liouville problem.

Assume: $\phi_n(x)$, λ_n can be computed.

University of Idaho

Solution: $u(x,t) = \phi_n(x)\, e^{-\lambda_n t}$

Principle of superposition:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x)\, e^{-\lambda_n t}$$

$\underline{IC}$: $u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$: generalized Fourier series

Orthogonality:

$$a_n = \frac{\int_0^L f(x)\,\phi_n(x)\,\sigma(x)\,dx}{\int_0^L \phi_n^2(x)\,\sigma(x)\,dx} = \frac{\int_0^L f(x)\,\phi_n(x)\,c(x)\,\rho(x)\,dx}{\int_0^L \phi_n^2(x)\,c(x)\,\rho(x)\,dx}$$

This information comes from properties of the regular Sturm-Liouville problem.

$\underline{\underline{Q}}$ What happens with the solution as $t \to \infty$?

University of Idaho

Because

$$\lambda_1 < \lambda_2 < \dots \quad < \lambda_n < \lambda_n < \dots$$

and

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}$$

λ_1 will be the dominant term and

$$u(x,t) \approx a_1 \phi_1(x) e^{-\lambda_1 t}$$