Lecture 28

Ex Heat flow in a nonuniform rod

$c\rho u_t = (K_0 u_x)_x$ in $0 < x < L$, $t > 0$

$u(0,t) = 0$

$u_x(L,t) = 0$

$u(x,0) = f(x)$

Separation of variables: $u(x,t) = \phi(x)\, h(t)$

$$c(x)\rho(x)\,\phi(x)\,\frac{dh}{dt} = h(t)\,\frac{d}{dx}\left(K_0(x)\,\frac{d\phi}{dx}\right) \quad \Big| \; \frac{1}{c(x)\rho(x)\,\phi(x)\,h(t)}$$

$$\frac{\frac{dh}{dt}}{h(t)} = \frac{\frac{d}{dx}\left(K_0(x)\,\frac{d\phi}{dx}\right)}{c(x)\rho(x)\,\phi(x)} = -\lambda$$

$$\frac{dh}{dt} + \lambda h = 0 \quad \Rightarrow \quad h(t) = C e^{-\lambda t}$$

$$\begin{cases} \dfrac{d}{dx}\left(K_0(x)\dfrac{d\phi}{dx}\right) + \lambda c(x)\rho(x)\phi(x) = 0 \\ \phi(0) = 0, \quad \dfrac{d\phi}{dx}(L) = 0 \end{cases}$$

This is a regular Sturm-Liouville problem with

$p(x) = K_0(x) > 0$, $q(x) = 0$, $\sigma(x) = c(x)\rho(x) > 0$, $a = 0$, $b = L$

Assume: $\phi_n(x)$, λ_n can be computed

Solution: $u(x,t) = \phi_n(x) e^{-\lambda_n t}$

Principle of linear superposition:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}$$

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IC: at $t=0$ $\quad u(x,0) = \sum_{n=1}^{\infty} a_n \Phi_n(x) = f(x)$

this is a generalized Fourier series

Orthogonality:

$$a_n = \frac{\int_0^L f(x)\, \Phi_n(x)\, \sigma(x)\, dx}{\int_0^L \Phi_n^2(x)\, \sigma(x)\, dx} = \frac{\int_0^L f(x)\, \Phi_n(x)\, c(x)\, \rho(x)\, dx}{\int_0^L \Phi_n^2(x)\, c(x)\, \rho(x)\, dx}$$

This information comes from properties of the regular Sturm-Liouville problem.

Q What happens with the solution as $t \to \infty$?

We showed previously that e'values $\lambda_n > 0$ strictly.
Moreover we know

$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1} < \dots$$

and

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}$$

So, λ_1 is the smallest e'value and the first term will be the dominant term, so

$$u(x,t) \approx a_1 \phi_1(x) e^{-\lambda_1 t}$$

Let us show $a_1 \neq 0$ and $\lambda_1 > 0$.

Note $c(x) > 0$, $\rho(x) > 0$ and let us assume $f(x) > 0$ (initial temperature)

From orthogonality:

$$a_1 = \frac{\displaystyle\int_0^L \overbrace{f(x)}^{>0} \phi_1(x) \overbrace{c(x)}^{>0} \overbrace{\rho(x)}^{>0} \, dx}{\displaystyle\int_0^L \phi_1^2(x) \underbrace{c(x)}_{>0} \underbrace{\rho(x)}_{>0} \, dx} \quad > 0$$

$\phi_1(x)$ must remain of the same sign because $\phi_1(x)$ has $n-1\big|_{n=1} = 0$ roots in $(0, L)$, i.e. no roots in $(0, L)$

denominator > 0, numerator is either > 0 or < 0

$\therefore \quad a_1 \neq 0$

Rayleigh Quotient:

$$\lambda = \frac{-p\phi\phi'\Big|_a^b + \int_a^b \left[p(\phi')^2 - \cancelto{0}{q\phi^2}\right]dx}{\int_a^b \phi^2 \sigma \, dx} = \frac{\cancelto{0}{-K_0(x)\phi\phi'\Big|_0^L} + \int_0^L K_0(\phi')^2 dx}{\int_0^L \phi^2 c\rho \, dx}$$

$p(x) = K_0(x)$, $q(x) = 0$, $\sigma(x) = c(x)\rho(x)$

$a = 0$, $b = L$

$$\underbrace{\phi(0) = 0, \quad \phi'(L) = 0}_{\Downarrow}$$

$$\left[-K_0(x)\phi\phi'\right]\Big|_{x=0}^{x=L} = 0$$

$$\therefore \quad \lambda = \frac{\int_0^L \overset{\geq 0}{K_0(x)} \overset{\geq 0}{(\phi')^2} dx}{\int_0^L \phi^2 c\rho \, dx}_{>0} \geq 0 \quad \Rightarrow \text{ all e'values } \lambda_n \geq 0$$

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Check if $\lambda = 0$ is an e'value

$\lambda = 0$ is an e'value $\Rightarrow \lambda = \dfrac{\int_0^L K_0(x)(\phi')^2 dx}{\int_0^L \phi^2 c \rho \, dx} = 0$

$\Rightarrow \int_0^L \overbrace{K_0(x)}^{>0} \overbrace{(\phi')^2}^{\geq 0} dx = 0 \Rightarrow \phi' \equiv 0 \Rightarrow \phi \equiv \text{const}$

$\phi(0) = 0, \quad \phi'(L) = 0$

$\Downarrow$

$\phi \equiv 0$ trivial solution $\lightning$ $\lambda = 0$ is not an e'value

Hence,

$$\lambda_n > 0 \qquad n = 1, 2, \ldots$$

in particular $\lambda_1 > 0$, this is a dominant e'value and indeed

$$u(x,t) \approx a_1 \phi_1(x) e^{-\lambda_1 t} \quad \text{and} \quad u(x,t) \xrightarrow[t \to \infty]{} 0$$

Solving non-constant coefficient ODEs

Consider a general 2^{nd} order linear homogeneous differential equation w/ variable coefficients:

$$y''(x) + a(x)\,y'(x) + b(x)\,y(x) = 0$$

How can we solve this equation?

1. If $a(x) = \text{const}$, $b(x) = \text{const}$ $\Rightarrow$ assume $y(x) = e^{rx}$
2. If $b(x) = 0$, then solve 1^{st} order ODE for $y'(x)$ (substitution $v(x) = y'(x)$).