Lecture 29

Last time we showed that

$$u(x,t) \approx a_1 \phi_1(x) e^{-\lambda_1 t}$$

Let us show that $a_1 \neq 0$ and $\lambda_1 > 0$

<u>Note</u>: $c(x) > 0$, $\rho(x) > 0$ and assume that $f(x) > 0$ (initial temperature).

$\phi_1(x)$ must remain ~~of~~ the same sign because the mode with $n=1$ has $n-1$ roots on $(0,L)$, i.e. no roots on $(0,L)$

From orthogonality:

$$a_1 = \frac{\int_0^L \overset{>0}{f(x)}\, \overset{>0}{\phi_1(x)}\, \overset{>0}{c(x)}\, \rho(x)\, dx}{\underbrace{\int_0^L \phi_1^2(x) \underbrace{c(x)}_{>0} \underbrace{\rho(x)}_{>0}\, dx}_{>0}} \neq 0$$

$\therefore \quad a_1 \neq 0$

Rayleigh Quotient:

$$\lambda = \frac{\cancel{-p\phi\phi'\Big|_a^b}^{\,0} + \int_a^b \left[p(\phi')^2 - \cancel{q}^{\,0}\phi^2 \right] dx}{\int_a^b \phi^2 \sigma \, dx}$$

$p(x) = K_0(x)$, $q(x) = 0$, $\sigma(x) = c(x)\rho(x)$ $\quad a = 0, \quad b = L$

$\phi(0) = 0$, $\phi'(L) = 0$ $\Rightarrow$ $\left[- K_0(x)\phi(x)\phi'(x) \right]\Big|_0^L = 0$

$$\therefore \quad \lambda = \frac{\int_0^L \overset{>0}{K_0(x)} \overset{\geq 0}{(\phi')^2} dx}{\underbrace{\int_0^L \underbrace{\phi^2}_{\geq 0} \underbrace{c(x)}_{>0} \underbrace{\rho(x)}_{>0} dx}_{>0}} \geq 0$$

Check if $\lambda = 0$ is an e'value.

$\lambda = 0$ is an e'value $\Rightarrow$ $\int_0^L \underbrace{K_0(x)}_{>0} (\phi')^2 dx = 0$

$\Rightarrow \phi'(x) \equiv 0 \Rightarrow \phi(x) \equiv \text{const}$

BC: $\phi(0)=0 \Rightarrow const = 0 \Rightarrow \phi(x) \equiv 0$ ↯

$\Rightarrow \lambda > 0$, i.e. all e'values are positive $\Rightarrow \lambda_1 > 0$ and gives dominant contribution since λ_1 is the smallest e'value

$$\therefore \quad u(x,t) \approx \underset{\neq 0}{a_1} \underbrace{\phi_1(x)}_{\not\equiv} e^{-\overset{>0}{\lambda_1} t} \quad \text{and} \quad u(x,t) \longrightarrow 0 \text{ as } t \to \infty$$

Solving non-constant coefficient ODEs

Consider a general 2^{nd} order linear homogeneous differential eq^n w/ variable coefficients:

$$y''(x) + a(x)y'(x) + b(x)y = 0$$

How can we solve this equation?

1. If $a(x) = const$, $b(x) = const \Rightarrow$ assume solution is the form $y(x) = e^{rx}$

2. If $b(x)=0$, then solve a 1st order ODE for $y'(x)$ (substitution $v(x)=y'(x)$).

3. If $a(x)=\frac{c}{x}$, $b(x)=\frac{d}{x^2}$

$$y''+\frac{c}{x}y'+\frac{d}{x^2}y=0 \quad \Big/ \cdot x^2$$

$$x^2y''+cxy'+dy=0$$: Euler or equidimensional eq^{n}

Assume solution in the form

$$y(x)=x^r$$

4. If $a(x)$, $b(x)$ are polynomials in x:

$y(x)=\sum\limits_{n=1}^{\infty} a_n x^n$: power series solution

$$y'=\sum_{n=1}^{\infty} a_n\cdot n x^{n-1}, \qquad y''=\sum_{n=1}^{\infty} a_n\cdot n(n-1)x^{n-2}$$

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Derive a recursion relation for a_n's and solve

5. If $a(x)$, $b(x)$ are general functions of x?

Option #1 : solve numerically (Math 428)

Option #2 : special functions : you may be able to write your solution in terms of special functions that cannot be evaluated explicitly but have special properties

Ex $y'' + xy = 0$

Solution can be written in terms of Airy functions that are complicated integrals that cannot be computed in closed form but tables of their numerical values are available. Asymptotic behaviour is also known.

Ex $x^2 y'' + x y' + (x^2 - n^2) y = 0$ Bessel's equation of order n

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Self-Adjoint Operators and Sturm-Liouville Problems

Recall the regular Sturm-Liouville problem

(1) $$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0 \qquad a < x < b$$

$$\beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) = 0 \qquad \beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) = 0$$

$p(x)$, $q(x)$, $\sigma(x)$: real-valued, continuous and $p(x)>0$, $\sigma(x)>0$ on $[a,b]$

$\beta_1, \beta_2, \beta_3, \beta_4$: real numbers

Operator Notation

Let $\mathcal{L} = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$: Sturm-Liouville operator i.e. $\mathcal{L}u = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x)$

eq^{n}(1) can be written as $\Rightarrow$ $\boxed{\mathcal{L}(\phi) + \lambda\sigma\phi = 0}$

Lagrange Identity

$\mathcal{L}(u) = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) \quad | \cdot v$

$-$

$\mathcal{L}(v) = \frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + q(x)v(x) \quad | \cdot u$

$u\,\mathcal{L}(v) - v\,\mathcal{L}(u) = u\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + \cancel{qvu} - v\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)$

$- \cancel{quv}$ $=$

Recall

$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \frac{df}{dx}\cdot g(x) + f(x)\cdot\frac{dg}{dx}$

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$$\Rightarrow f(x)\frac{dg}{dx} = \frac{d}{dx}\big(f(x)\cdot g(x)\big) - \frac{df}{dx}\cdot g(x)$$

$\therefore$

$$u\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) = \frac{d}{dx}\left(u(x)\,p(x)\frac{dv}{dx}\right) - \frac{du}{dx}\cdot p(x)\frac{dv}{dx}$$

Similarly,

$$v\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) = \frac{d}{dx}\left(v(x)\,p(x)\frac{du}{dx}\right) - \frac{dv}{dx}\cdot p(x)\frac{du}{dx}$$

$$= \frac{d}{dx}\left(u(x)p(x)\frac{dv}{dx}\right) - \cancel{\frac{du}{dx}\cdot p(x)\frac{dv}{dx}} - \frac{d}{dx}\left(v(x)p(x)\frac{du}{dx}\right) + \cancel{\frac{dv}{dx}p(x)\frac{du}{dx}}$$

$$= \frac{d}{dx}\left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]$$

Then

$$\boxed{u\mathcal{L}(v) - v\mathcal{L}(u) = \frac{d}{dx}\left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]} :$$

Lagrange identity in a differential form

Integrate $\int_a^b$

$$\boxed{\int_a^b \left[u\,\mathcal{L}(v) - v\,\mathcal{L}(u)\right] dx = \left\{ p(x) \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \right\} \Big|_a^b}$$

<u>Lagrange identity in integral form</u>

or <u>Green's formula</u>