Lecture 29

Solving non-constant coefficient ODEs

(Cont'd)

3. If $a(x) = \frac{c}{x}$, $b(x) = \frac{d}{x^2}$

$$y'' + \frac{c}{x} y' + \frac{d}{x^2} y = 0 \quad \Big| \cdot x^2$$

$x^2 y'' + c x y' + d y = 0$: Euler or equidimensional eq^n

Assume solution in the form $y(x) = x^r$

4. If $a(x)$, $b(x)$ are polynomials in x:

$y(x) = \sum_{n=1}^{\infty} a_n x^n$: power series solution

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute into DE and find recursive relations for a_n's, and solve for a_n's.

5. If $a(x)$, $b(x)$ are general functions of x?

<u>Option #1</u>: solve numerically (Math 428)

<u>Option #2</u>: use special functions: you may be able to write your solution in terms of special functions. You may not be able to evaluate your solution directly but you will be able to use properties of these special functions.

<u>Ex</u> $y'' + xy = 0$

Solution can be written in terms of <u>Airy</u>

functions. These functions are complicated integrals that cannot be computed in closed form but tables of numerical values are available. Asymptotic behaviour is also available.

Ex $x^2y'' + xy' + (x^2-n^2)y = 0$ Bessel equation of order n

Self-Adjoint Operators and Sturm-Liouville problems

Recall that the regular Sturm-Liouville problem is

(1) $$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0 \qquad a < x < b$$

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0 \qquad \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0$$

$p(x), q(x), \sigma(x)$: real-valued, continuous & $p(x)>0$, $\sigma(x)>0$

$\beta_1, \beta_2, \beta_3, \beta_4$: real #s

Operator Notation

Let

$$\mathcal{L} = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x):$$ Sturm-Liouville operator

i.e.

$$\mathcal{L}u = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x)$$

Then S-L. equation (1) can be written as

$$\boxed{\mathcal{L}(\phi) + \lambda\sigma\phi = 0}$$

Lagrange Identity

$$\mathcal{L}(u) = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)\,u(x) \qquad \Big|\cdot v$$

$$-$$

$$\mathcal{L}(v) = \frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + q(x)\,v(x) \qquad \Big|\cdot u$$

$$v\,\mathcal{L}(u) - u\,\mathcal{L}(v) = v\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + \cancel{q\,u\,v} -$$

$$- \cancel{q\,v\,u} - u\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) =$$

Recall

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \frac{df}{dx}\cdot g(x) + f(x)\frac{dg}{dx}$$

$$\Rightarrow \quad f(x)\frac{dg}{dx} = \frac{d}{dx}\left(f(x)\cdot g(x)\right) - \frac{df}{dx}\cdot g(x)$$

$$\therefore \quad v\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] = \frac{d}{dx}\left(v(x)p(x)\frac{dy}{dx}\right) - \frac{dv}{dx}\cdot p(x)\frac{dy}{dx}$$

similarly,

$$u\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right] = \frac{d}{dx}\left(u(x)p(x)\frac{dv}{dx}\right) - \frac{du}{dx}\cdot p(x)\frac{dv}{dx}$$

$$= \frac{d}{dx}\left(v(x)p(x)\frac{dy}{dx}\right) - \cancel{\frac{dv}{dx}\cdot p(x)\frac{dy}{dx}} - \frac{d}{dx}\left(u(x)p(x)\frac{dv}{dx}\right)$$

$$+ \cancel{\frac{du}{dx}p(x)\frac{dv}{dx}} = \frac{d}{dx}\left(p(x)\left[v(x)\frac{dy}{dx} - u(x)\frac{dv}{dx}\right]\right)$$

Then

$$\boxed{u\,\mathcal{L}(v) - v\,\mathcal{L}(u) = \frac{d}{dx}\left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]}$$

<u>Lagrange identity</u> in a <u>differential form</u>

Integrate $\int_a^b$:

$$\boxed{\int_a^b \left[u\,\mathcal{L}(v) - v\,\mathcal{L}(u)\right] dx = \left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]_{x=a}^{x=b}} \quad (2)$$

<u>Lagrange identity</u> in an <u>integral form</u> or

<u>Green's formula</u>

Now let's consider Sturm-Liouville operator w/ BCs:

$$\beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) = 0 \qquad \beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) = 0$$

let u, v satisfy the same BCs:

$$\beta_1 u(a) + \beta_2 \frac{du}{dx}(a) = 0 \qquad \beta_1 v(a) + \beta_2 \frac{dv}{dx}(a) = 0$$

$$\beta_3 u(b) + \beta_4 \frac{du}{dx}(b) = 0 \qquad \beta_3 v(b) + \beta_4 \frac{dv}{dx}(b) = 0$$

We want to show that the boundary term in Green's formula (2) vanishes.

$$\left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]_{x=a}^{x=b} = p(b)\left(u(b)\frac{dv}{dx}(b) - v(b)\frac{du}{dx}(b)\right) -$$

$$- p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right)$$

at $x=a$: $\quad p(a)\left(u(a) \underbrace{\frac{dv}{dx}(a)}_{= -\frac{\beta_1}{\beta_2} v(a)} - v(a) \underbrace{\frac{du}{dx}(a)}_{= -\frac{\beta_1}{\beta_2} u(a)} \right) = 0$

from BCs

similarly, boundary term at $x=b$ also vanishes.

$$\therefore \quad \boxed{\int_a^b \left[u\,\mathcal{L}(v) - v\,\mathcal{L}(u) \right] dx = 0}$$

<u>Def</u> An operator $\mathcal{L}$ with corresponding BCs is <u>self-adjoint</u> if

$$\int_a^b \left[u\,\mathcal{L}(v) - v\,\mathcal{L}(u) \right] dx = 0$$

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where $u(x)$ and $v(x)$ satisfy the same BCs.

<u>Note</u> We showed above that Sturm-Liouville operator with BCs from regular Sturm-Liouville problem is self-adjoint.

<u>Ex</u> Periodic BCs. (not a regular S.-L. problem)

$$\phi(a) = \phi(b) \qquad p(a)\,\phi'(a) = p(b)\,\phi'(b)$$

$$\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]_{x=a}^{x=b} = p(b)\left(\overset{u(a)}{u(b)}\frac{dv}{dx}(b) - \overset{v(a)}{v(b)}\frac{du}{dx}(b)\right)$$

$p(a)\frac{dv}{dx}(a)$

$$-p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right) =$$

$$= p(a)\frac{du}{dx}(a)$$

$$= u(a)\,\cancel{p(a)\frac{dv}{dx}(a)} - \cancel{v(a)\,p(a)\frac{dy}{dx}(a)} - p(a)\Big(\cancel{u(a)\frac{dv}{dx}(a)} - \cancel{v(a)\frac{dy}{dx}(a)}\Big) = 0$$

Periodic BCs;

$u(a) = u(b)$ $\quad$ $v(a) = v(b)$

$p(a)\,u'(a) = p(b)\,u'(b)$ $\quad$ $p(a)\,v'(a) = p(b)\,v'(b)$

$\therefore$ Sturm-Liouville problem w/ periodic BCs is also self-adjoint.