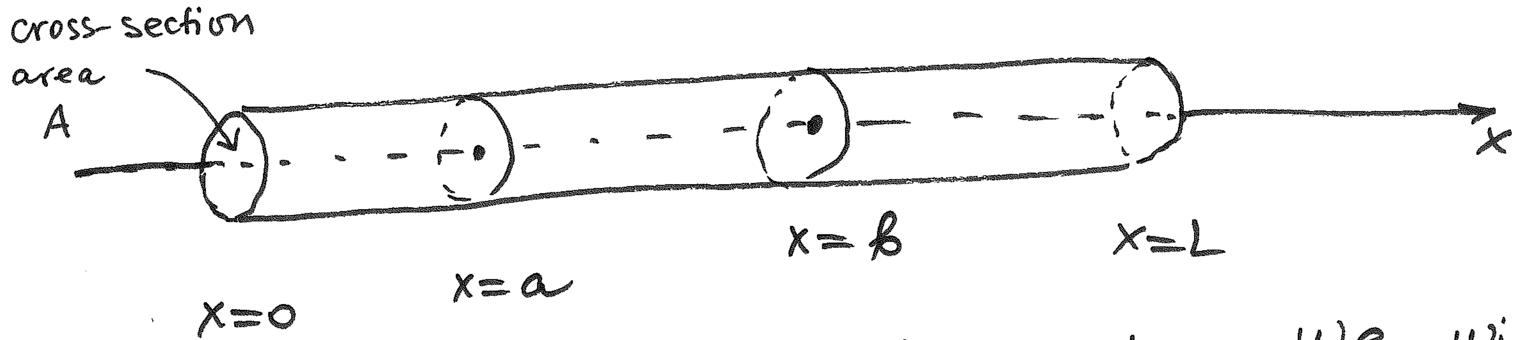
Lecture 2

Heat Equation

1.2 Heat equation in 1D rod



Consider a 1D rod of length L. We will derive an equation that governs the distribution of temperature inside the rod.

We will use the conservation of energy principle. In this case, energy is the thermal energy.

Assume that length L of the rod is much bigger than the rod thickness. We can also assume that the energy depends only on x and time t, but there is no dependence on y variable / direction.

A: cross-section area; assume $A = \text{const}$

Let $e(x,t)$ be the thermal energy density per unit volume.

<u>Q1</u> What is the total thermal energy of the rod between $x=a$ and $x=b$?

$$\int_a^b e(x,t) \underbrace{A\,dx}_{\text{volume element}}$$

Q2 How can the total energy be changed?

We assume that the lateral surface of the rod is insulated.

1. <u>Heat flux through the boundary</u>

$\phi(x,t)$: amount of thermal energy per unit time that flows to the right through unit surface

2. <u>Heat energy generated inside the rod due to sources/sinks</u>

$Q(x,t)$: thermal energy generated inside the rod due to sources/sinks per unit time per unit volume

Heat flow process (conservation of thermal energy):

rate of change of the total thermal energy	=	heat energy that flows through boundaries per unit time	+	total energy due to sources/sinks inside the rod per unit time

$$\frac{d}{dt}\int_a^b e(x,t)A\,dx = \phi(a,t)A - \phi(b,t)A + \int_a^b Q(x,t)A\,dx$$

Cancel A ($A = \text{const}$)

$$\boxed{\frac{d}{dt}\int_a^b e(x,t)\,dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)\,dx}$$

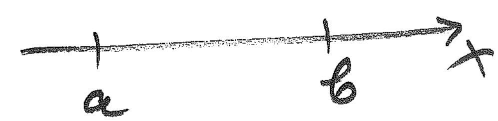
conservation of energy in integral form

If $e(x,t)$ is differentiable, then

$$\frac{d}{dt}\int_a^b e(x,t)\,dx = \int_a^b \frac{\partial e(x,t)}{\partial t}\,dx$$

If $\phi(x,t)$ is continuously differentiable, then by the Fundamental Thm of Calculus,

$$\phi(a,t) - \phi(b,t) = -\int_a^b \frac{\partial \phi}{\partial x}\,dx$$



$$\Rightarrow \int_a^b \frac{\partial e}{\partial t}\,dx = -\int_a^b \frac{\partial \phi}{\partial x}\,dx + \int_a^b Q(x,t)\,dx$$

$$\therefore \boxed{\int_a^b \left[\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q\right] dx = 0}$$

Since a, b were arbitrary points between 0 and L,

$$(1) \quad \boxed{\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q = 0}$$

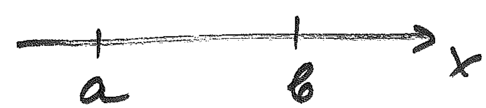
conservation of energy law in differential form

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$$

let $Q=0 \Rightarrow \frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$

If $\frac{\partial \phi}{\partial x} > 0 \Rightarrow \phi$ increases with x

$\Rightarrow \phi(a,t) < \phi(b,t)$



$$\phi(a,t) = \phi(x,t)\Big|_{x=a} \qquad \phi(b,t) = \phi(x,t)\Big|_{x=b}$$

$\frac{\partial e}{\partial t} = -\underbrace{\frac{\partial \phi}{\partial x}}_{>0} < 0 \Rightarrow e(x,t)$ should decrease in time, i.e. thermal energy decreases in time

Q What is the relation between thermal density $e(x,t)$ and temperature?

$$e(x,t) = c(x)\,\rho(x)\,u(x,t)$$

$c(x)$: specific heat (amount of energy needed to increase temperature of a unit mass by one unit)

$\rho(x)$: mass density (per unit volume)

Then eq^{n} (1) can be written as

$$\boxed{c\rho \frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} - Q = 0} \qquad (2)$$

Fourier's Law of Heat Conduction

$$\phi = -K_0 \frac{\partial u}{\partial x}$$

K_0: thermal conductivity (measures the ability of material to conduct heat)

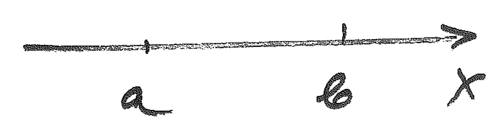
Properties of heat flow:

1. If the temperature is the same everywhere, there is no heat flow.
2. If there is temperature variations, then heat flows from hot area to cold.
3. The higher temperature differences $\Rightarrow$ more intensive heat flow.
4. Heat flow is material dependent.

$\phi = -K_0 \frac{\partial u}{\partial x}$: Fourier Law of heat conduction

Let $\frac{\partial u}{\partial x} > 0 \Rightarrow u \nearrow$ to the right $\Rightarrow u\Big|_{x=a} < u\Big|_{x=b}$

$\Rightarrow$ heat flows to the left



We define heat flux positive if heat flows to the right.

$$\phi = -\underbrace{K_0}_{>0} \underbrace{\frac{\partial u}{\partial x}}_{>0} < 0$$ $\Rightarrow$ this explains the minus sign

Substitute this into (2)

$$c\rho \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(-K_0 \frac{\partial u}{\partial x}\right) - Q = 0$$

$$\boxed{c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q} :$$

2nd order PDE
linear, nonhomog. if $Q \neq 0$

or

$$\boxed{c\rho \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) = Q}$$

general heat conduction equation

Assume the material properties are const :

$c = \text{const}$ $\quad \rho = \text{const}$ $\quad K_0 = \text{const}$

let $Q = 0$

$$\therefore \quad \boxed{\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}}$$

heat eqn w/ constant material properties

where $K = \dfrac{K_0}{c\rho}$: thermal diffusivity