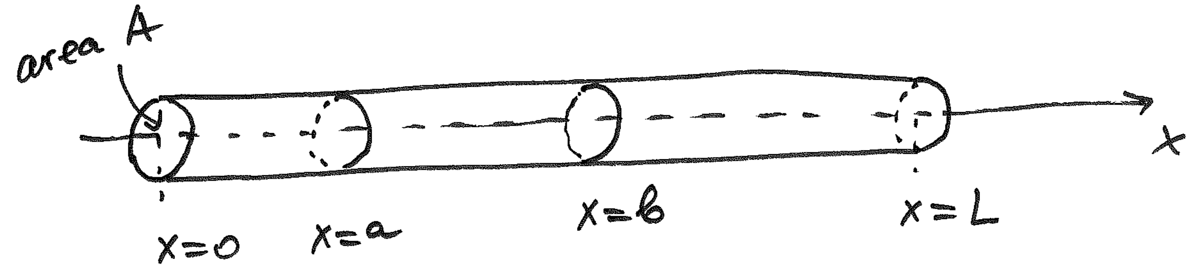
Heat Equation

1.2 Heat equation in 1D rod

Consider a 1D rod of length L



We want to derive an equation that governs the temperature distribution inside the rod.

We will use the "conservation of energy" principle. In this case, the energy is the thermal energy.

Assume that the rod has length L much bigger than its thickness. Then we can assume that energy depends only on x and t, but there is no variation in y direction.

Denote by A the cross-section area of the rod and assume $A = \text{const}$.

Let $e(x,t)$ be the thermal energy density per unit volume.

<u>Q1</u> What is the total thermal energy of the rod between $x = a$ and $x = b$?

$$\int_a^b e(x,t)\, A\, dx$$

<u>Q2</u> How can the total energy be changed?

We assume that the lateral surface of the rod is insulated.

1. <u>Heat flux through boundary</u>

$\Phi(x,t)$: amount of thermal energy per unit time per unit surface that flows to the right through

2. Heat energy generated inside the rod due to sources/sinks

$Q(x,t)$: thermal energy generated inside the rod due to sources/sinks per unit time per unit volume

Heat flow process:

rate of change of the total thermal energy = heat energy that flows through boundaries per unit time + total energy due to sources/sinks, generated inside the rod per unit time

$$\frac{d}{dt}\int_a^b e(x,t)\,A\,dx = \phi(a,t)A - \phi(b,t)\cdot A + \int_a^b Q(x,t)\,A\,dx$$

Cancel by A

$$\boxed{\frac{d}{dt}\int_a^b e(x,t)\,dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)\,dx}$$

conservation of energy in integral form

If $e(x,t)$ is differentiable, then

$$\frac{d}{dt}\int_a^b e(x,t)\,dx = \int_a^b \frac{\partial e(x,t)}{\partial t}\,dx$$

If $\phi(x,t)$ is continuously differentiable, then by the Fundamental Thm of Calculus

$$\phi(a,t)-\phi(b,t) = -\int_a^b \frac{\partial \phi}{\partial x}\,dx$$

$$\Rightarrow \int_a^b \frac{\partial e}{\partial t}\,dx = -\int_a^b \frac{\partial \phi}{\partial x}\,dx + \int_a^b Q(x,t)\,dx$$

$$\therefore \boxed{\int_a^b \left[\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q\right] dx = 0}$$

conservation of energy law in integral form

Since a, b were arbitrary values between 0 and L,

(1) $$\boxed{\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q = 0}$$ conservation of energy law in differential form

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$$

let $Q = 0$

If $\frac{\partial \phi}{\partial x} > 0 \Rightarrow \phi$ increases with x

$$\Rightarrow \phi(a,t) < \phi(b,t)$$

$$\phi(a,t) = \phi(x,t)\Big|_{x=a} \qquad \phi(b,t) = \phi(x,t)\Big|_{x=b}$$

$$\Rightarrow \frac{\partial e}{\partial t} = -\underbrace{\frac{\partial \phi}{\partial x}}_{>0} < 0$$ $\Rightarrow$ $e(x,t)$ should decrease in time, i.e. thermal energy decreases in time

Q What is the relation between thermal energy $e(x,t)$ and temperature $u(x,t)$?

$$e(x,t) = c(x)\,\rho(x)\,u(x,t)$$

$c(x)$: specific heat (amount of energy needed to increase temperature of a unit mass by one unit)

$\rho(x)$: mass density (mass per unit volume)

Eq^{n} (1) gives

$$\boxed{c\rho\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} - Q = 0} \qquad (2)$$

Fourier's Law of Heat Conduction

Properties of the heat flow:

1. If the temperature is the same everywhere, there is no heat flow.

2. If there is temperature variation, then heat flows from hot area to cold.
3. The higher temperature differences $\Rightarrow$ more intensive heat flow
4. Heat flow is material dependent.

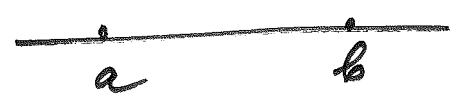
$$\boxed{\phi = -K_0 \frac{\partial u}{\partial x}}$$

heat flux is proportional to the temperature difference

$K_0 = K_0(x)$: <u>thermal conductivity</u> (measures the ability of material to conduct heat)

Let $\frac{\partial u}{\partial x} > 0 \Rightarrow u \nearrow$ to the right $\Rightarrow u\big|_{x=a} < u\big|_{x=b}$

$\Rightarrow$ heat flows to the left



$$\phi = -\underbrace{K_0}_{>0} \underbrace{\frac{\partial u}{\partial x}}_{>0} < 0$$

this explains the "$-$" sign

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Substitute $\phi = -K_0 \frac{\partial u}{\partial x}$ into Eq. (2).

$$c\rho \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(-K_0 \frac{\partial u}{\partial x}\right) - Q = 0$$

$$\boxed{c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q}$$

2nd order PDE

linear, nonhomog. if $Q \neq 0$

doesn't depend on u

$$\boxed{c\rho \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) = Q}$$

General heat conduction equation

Assume that all material properties are constant:

$c = \text{const}$, $\rho = \text{const}$, $K_0 = \text{const}$ and let $Q = 0$

$$\therefore \quad \boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}}$$

heat equation w/ const material properties

$k = \frac{K_0}{c\rho}$: thermal diffusivity

Initial and Boundary Conditions

We have to prescribe an initial condition / initial temperature distribution:

$$u(x,0) = f(x): \quad \underline{\text{initial condition}}$$