Lecture 30

Self-Adjoint Operators

Consider Sturm-Liouville operator $\mathcal{L}$ <u>with</u> BCs:

$\beta_1 \phi(a) + \beta_2 \phi'(a) = 0 \qquad \beta_3 \phi(b) + \beta_4 \phi'(b) = 0$

Let u, v satisfy the same BCs:

$\beta_1 u(a) + \beta_2 u'(a) = 0 \qquad \beta_1 v(a) + \beta_2 v'(a) = 0$

$\beta_3 u(b) + \beta_4 u'(b) = 0 \qquad \beta_3 v(b) + \beta_4 v'(b) = 0$

We want to show that the boundary term in Green's formula (*) vanishes.

$$\int_a^b \left[u\mathcal{L}(v) - v\mathcal{L}(u)\right] dx = \left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]\Bigg|_a^b \qquad (*)$$

<u>Green's formula</u>

$$\left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]\Bigg|_a^b = p(b)\left(u(b)\frac{dv}{dx}(b) - v(b)\frac{du}{dx}(b)\right)$$

$$- p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right)$$

at $x=a$: $$p(a)\Big(u(a)\underbrace{\frac{dv}{dx}(a)}_{= -\frac{\beta_1}{\beta_2}v(a)} - v(a)\underbrace{\frac{du}{dx}(a)}_{-\frac{\beta_1}{\beta_2}u(a)}\Big) = 0$$

from BCs

Similarly, boundary term at $x=b$ vanishes.

$$\therefore \quad \boxed{\int_a^b \left[u\,\mathcal{L}(v) - v\,\mathcal{L}(u)\right] dx = 0}$$

<u>Def</u> An operator $\mathcal{L}$ with corresponding BCs is <u>self-adjoint</u>

if $$\int_a^b \left[u\,\mathcal{L}(v) - v\,\mathcal{L}(u)\right] dx = 0$$

where $u(x)$ and $v(x)$ satisfy the same BCs.

Note We showed above that Sturm-Liouville operator with BCs from regular Sturm-Liouville problem is self-adjoint.

S-L

Ex Periodic BCs (not regular S-L problem)

$\phi(a) = \phi(b)$ $\quad\quad$ $p(a)\phi'(a) = p(b)\phi'(b)$

$$\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]\Big|_a^b = p(b)\Big(\underbrace{u(b)}_{=u(a)}\frac{dv}{dx}(b) - \underbrace{v(b)}_{=v(a)}\frac{du}{dx}(b)\Big)$$

$p(a)\frac{dv}{dx}(a)$

$$-p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right) =$$

: $p(b)\frac{du}{dx}(b) = p(a)\frac{du}{dx}(a)$

Periodic BCs:

$u(a) = u(b)$ $\quad\quad$ $v(a) = v(b)$

$p(a)u'(a) = p(b)u'(b)$ $\quad\quad$ $p(a)v'(a) = p(b)v'(b)$

$$= u(a)p(a)\frac{dv}{dx}(a) - v(a)p(a)\frac{du}{dx}(a) - p(a)u(a)\frac{dv}{dx}(a) + p(a)v(a)\frac{du}{dx}(a) = 0$$

$\therefore$ Sturm-Liouville operator w/ periodic BCs is self-adjoint.

<u>Claim</u> Sturm-Liouville eigenfunctions corresponding to distinct e'values are orthogonal w/ weight σ.

□ Let ϕ_n, ϕ_m be two e'functions that correspond to e'values λ_n, λ_m, $\lambda_n \neq \lambda_m$.

$$\mathcal{L}(\phi_n) + \lambda_n \sigma \phi_n = 0 \qquad \mathcal{L}(\phi_m) + \lambda_m \sigma \phi_m = 0$$

$$\Downarrow \qquad \Downarrow$$

$$\mathcal{L}(\phi_n) = -\lambda_n \sigma \phi_n \qquad \mathcal{L}(\phi_m) = -\lambda_m \sigma \phi_m$$

S.-L. operator $\mathcal{L}$ is self-adjoint $\xRightarrow{\text{Green's}}$ formula

$$\int_a^b \Big[\phi_n \underbrace{\mathcal{L}(\phi_m)}_{= -\lambda_m \phi_m} - \phi_m \underbrace{\mathcal{L}(\phi_n)}_{-\lambda_n \phi_n}\Big]\, dx = 0$$

$$\int_a^b \left[\phi_n(-\lambda_m)\phi_m \sigma - \phi_m(-\lambda_n)\phi_n \sigma\right] dx = 0$$

$$\underbrace{(\lambda_n - \lambda_m)}_{\neq 0} \int_a^b \phi_n \phi_m \sigma \, dx = 0$$

since λ_n and λ_m are distinct $\Rightarrow \lambda_n - \lambda_m \neq 0$

$\therefore \int_a^b \phi_n \phi_m \sigma \, dx = 0, \quad n \neq m$

$\therefore$ ϕ_n, ϕ_m are orthogonal wrt weight σ on $[a, b]$. ■