Lecture 30

Self-Adjoint Operators (Cont'd)

Recall

$\mathcal{L} = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$: Sturm-Liouville operator

$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u + \lambda\sigma(x)u = 0$: Sturm-Liouville eq^{n}

$\mathcal{L}u + \lambda\sigma u = 0$ $\qquad$ $\mathcal{L}u$ or $\mathcal{L}(u)$

$\int_a^b [u\mathcal{L}v - v\mathcal{L}u]\,dx = \left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]_a^b$: Green's formula

Operator $\mathcal{L}$ w/ corresponding BCs is <u>self-adjoint</u> if

$\int_a^b [u\mathcal{L}v - v\mathcal{L}u]\,dx = 0$ $\qquad$ i.e. $\left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]_a^b = 0$

Claim Sturm-Liouville eigenfunctions corresponding to distinct e'values are orthogonal w/ weight σ.

□ Let ϕ_n, ϕ_m be two e'functions that correspond to e'values λ_n, λ_m, $\lambda_n \neq \lambda_m$.

$$\mathcal{L}(\phi_n) + \lambda_n \sigma \phi_n = 0 \qquad \mathcal{L}(\phi_m) + \lambda_m \sigma \phi_m = 0$$

$$\Downarrow \qquad \Downarrow$$

$$\mathcal{L}(\phi_n) = -\lambda_n \sigma \phi_n \qquad \mathcal{L}(\phi_m) = -\lambda_m \sigma \phi_m$$

S.-L. operator $\mathcal{L}$ is self-adjoint $\Rightarrow$ by Green's formula w/ $u = \phi_n$, $v = \phi_m$

$$\int_a^b \Big[\phi_n \underbrace{\mathcal{L}(\phi_m)}_{= -\lambda_m \sigma \phi_m} - \phi_m \underbrace{\mathcal{L}(\phi_n)}_{= -\lambda_n \sigma \phi_n}\Big]\, dx = 0$$

$$\int_a^b \left[-\lambda_m \sigma \phi_n \phi_m + \lambda_n \sigma \phi_m \phi_n \right] dx = 0$$

$$\underbrace{(\lambda_n - \lambda_m)}_{\neq 0} \int_a^b \phi_n \phi_m \sigma \, dx = 0$$

since $\lambda_n \neq \lambda_m$

$$\therefore \int_a^b \phi_n \phi_m \sigma \, dx = 0, \quad n \neq m$$

Hence, ϕ_n, ϕ_m are orthogonal w/ weight function $\sigma(x)$ on $[a, b]$. $\blacksquare$

<u>Claim</u> E'values are real.

□ Assume that e'value λ is complex and it has an associated e'function ϕ.

Recall $z = x + iy$, $\bar{z} = \underset{(eg^{\cdot})}{x - iy}$: complex conjugate of z

Sturm-Liouville problem is

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\,\phi(x) + \lambda\,\sigma(x)\,\phi(x) = 0 \qquad a \le x \le b$$

$p(x), q(x), \sigma(x)$: real-valued functions, $p(x), \sigma(x) > 0$

$$\mathcal{L} = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \quad \Rightarrow \quad \boxed{\mathcal{L}(\phi) + \lambda\sigma\phi = 0} \qquad (1)$$

Take a complex-conjugate of both sides of the eq^{n}.

$$\overline{\mathcal{L}(\phi)} + \bar{\lambda}\sigma\bar{\phi} = 0 \qquad (2)$$

$$\overline{\mathcal{L}(\Phi)} = \overline{\frac{d}{dx}\left(p(x)\frac{d\Phi}{dx}\right) + q(x)\,\Phi(x)} =$$

$$= \frac{d}{dx}\left(p(x)\frac{d\overline{\Phi}}{dx}\right) + q(x)\,\overline{\Phi}(x) = \mathcal{L}(\overline{\Phi}), \quad \text{ie. } \overline{\mathcal{L}(\Phi)} = \mathcal{L}(\overline{\Phi})$$

Then eq^{n} (2) can be written as

$$\boxed{\mathcal{L}(\overline{\Phi}) + \overline{\lambda}\,\sigma\,\overline{\Phi} = 0} \qquad (3)$$

BCs:

$$\boxed{\begin{aligned} &\beta_1\,\Phi(a) + \beta_2\,\frac{d\Phi}{dx}(a) = 0 \\ &\beta_3\,\Phi(b) + \beta_4\,\frac{d\Phi}{dx}(b) = 0 \end{aligned}} \qquad (4)$$

where $\beta_1, \dots, \beta_4$ are real #s. Hence

$$\boxed{\beta_1\,\overline{\Phi}(a) + \beta_2\,\frac{d\overline{\Phi}}{dx}(a) = 0, \qquad \beta_3\,\overline{\Phi}(b) + \beta_4\,\frac{d\overline{\Phi}}{dx}(b) = 0} \qquad (5)$$

equations (1), (4) and (3), (5) imply for ϕ and $\bar{\phi}$ that

λ is an e'value associated w/ e'function ϕ

$\bar{\lambda}$ is an e'value associated w/ e'function $\bar{\phi}$

We just showed that e'functions that correspond to distinct e'values are orthogonal w/ weight σ.

Orthogonality : $$(\lambda - \bar{\lambda}) \int_a^b \underbrace{\phi \bar{\phi}}_{>0} \underset{>0}{\sigma}\, dx = 0$$

<u>Note</u> $\phi = \phi_r + i\phi_i$, $\bar{\phi} = \phi_r - i\phi_i$

$$\phi \bar{\phi} = (\phi_r + i\phi_i)(\phi_r - i\phi_i) = \phi_r^2 + \phi_i^2$$

ϕ is an e'function $\Rightarrow \phi \not\equiv 0 \Rightarrow \phi_r$ or ϕ_i or both are $\neq 0$

$\Rightarrow \phi_r^2 + \phi_i^2 > 0 \Rightarrow \phi \bar{\phi} > 0$

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$\therefore \int_a^b \phi\bar{\phi}\,\sigma\,dx > 0 \Rightarrow \lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda}$ or

λ is real $\boxed{\text{E}}$

<u>Note</u>: e'functions ϕ_n also may be chosen to be real-valued.