Lecture 31

<u>Claim</u> E'values are real.

□ Assume that e'value λ is complex and it has a complex-valued associated e'function ϕ.

<u>Recall</u>: $z = x + iy$, $\bar{z} = x - iy$: complex conjugate of z

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0, \qquad a \le x \le b$$

$p(x), q(x), \sigma(x)$: real-valued functions, $p(x)>0$, $\sigma(x)>0$

$$\mathcal{L} = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \Rightarrow \boxed{\mathcal{L}(\phi) + \lambda\sigma\phi = 0} \qquad (*)$$

$$\Rightarrow \overline{\mathcal{L}(\phi)} + \bar{\lambda}\sigma\bar{\phi} = 0 \qquad (1)$$

$$\overline{\mathcal{L}(\phi)} = \overline{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi(x)} = \frac{d}{dx}\left(p(x)\frac{d\bar{\phi}}{dx}\right) + q(x)\bar{\phi}(x) =$$

$$= \mathcal{L}(\bar{\phi})$$

$\Rightarrow \overline{\mathcal{L}(\phi)} = \mathcal{L}(\bar{\phi})$

Then eq^{n} (1) can be written as

$$\boxed{\mathcal{L}(\bar{\phi}) + \bar{\lambda}\,\sigma\,\bar{\phi} = 0} \qquad (**)$$

BCs: $\beta_1 \phi(a) + \beta_2 \phi'(a) = 0$

$\beta_3 \phi(b) + \beta_4 \phi'(b) = 0$

where $\beta_1, \dots, \beta_4$ are real numbers. Hence,

$\beta_1 \bar{\phi}(a) + \beta_2 \bar{\phi}'(a) = 0$

$\beta_3 \bar{\phi}(b) + \beta_4 \bar{\phi}'(b) = 0$

Equations $(*)$ and $(**)$ plus BCs for ϕ and $\bar{\phi}$ imply that

λ is an e'value w/ associated e'function ϕ

$\bar{\lambda}$ is an e'value w/ associated e'function $\bar{\phi}$

We have shown that e'functions that correspond to distinct e'values are orthogonal (see Lecture 30)

Orthogonality:

$$(\lambda-\bar{\lambda})\int_a^b \underbrace{\phi\bar{\phi}}_{>0}\underbrace{\sigma}_{>0}\,dx = 0$$

Note: $\phi = \phi_r + i\,\phi_i$

$\bar{\phi} = \phi_r - i\,\phi_i$

$\phi\bar{\phi} = (\phi_r + i\,\phi_i)(\phi_r - i\,\phi_i) = \phi_r^2 + \phi_i^2 \geq 0$

ϕ is an e'function $\Rightarrow$ $\phi \not\equiv 0$ $\Rightarrow$ ϕ_r or ϕ_i (or both) are non-zero $\Rightarrow$ $\phi_r^2 + \phi_i^2 > 0$ $\Rightarrow$ $\phi\bar{\phi} > 0$

$\therefore$ $\int_a^b \phi\bar{\phi}\sigma\,dx > 0$ $\Rightarrow$ $\lambda - \bar{\lambda} = 0$ or $\lambda = \bar{\lambda}$ $\Rightarrow$ λ is real ∎

Note e'function $\phi(x)$ can also be chosen to be real-valued.

Claim Uniqueness of eigenfunctions (regular S.-L. problem):

eigenfunctions are defined up to an arbitrary non-zero multiplicative constant.

□

Assume that there are two different e'functions ϕ_1 and ϕ_2 that correspond to the same e'value λ.

$$\mathcal{L}(\phi_1) + \lambda\sigma\phi_1 = 0 \qquad /\ \phi_2$$

$$\mathcal{L}(\phi_2) + \lambda\sigma\phi_2 = 0 \qquad /\cdot \phi_1$$

$$\phi_2\mathcal{L}(\phi_1) + \cancel{\lambda\sigma\phi_1\phi_2} - \phi_1\mathcal{L}(\phi_2) - \cancel{\lambda\sigma\phi_1\phi_2} = 0$$

$$\phi_2\mathcal{L}(\phi_1) - \phi_1\mathcal{L}(\phi_2) = 0$$

Lagrange's identity:

$$0 = \int_a^b \left[\phi_2 \mathcal{L}(\phi_1) - \phi_1 \mathcal{L}(\phi_2) \right] dx = \frac{d}{dx}\left[p(\phi_2 \phi_1' - \phi_1 \phi_2') \right]$$

$$\Rightarrow \frac{d}{dx}\left[p(\phi_2 \phi_1' - \phi_1 \phi_2') \right] = 0$$

$$\Rightarrow p(\phi_2 \phi_1' - \phi_1 \phi_2') = \text{const} = p(a)\left(\phi_2(a)\phi_1'(a) - \phi_1(a)\phi_2'(a) \right) =$$

BCs at $x = a$

$$\beta_1 \phi_1(a) + \beta_2 \phi_1'(a) = 0 \qquad\qquad \beta_1 \phi_2(a) + \beta_2 \phi_2'(a) = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\phi_1'(a) = -\frac{\beta_1}{\beta_2}\phi_1(a) \qquad\qquad \phi_2'(a) = -\frac{\beta_1}{\beta_2}\phi_2(a)$$

$$= p(a)\left[\phi_2(a)\cdot\left(-\frac{\beta_1}{\beta_2}\right)\phi_1(a) - \phi_1(a)\cdot\left(-\frac{\beta_1}{\beta_2}\right)\phi_2(a) \right] = 0$$

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$$\therefore \quad \underbrace{p(x)}_{\substack{>0 \\ \neq 0}} \left[\phi_2 \phi_1' - \phi_1 \phi_2' \right] = 0$$

$$\Rightarrow \phi_2 \phi_1' - \phi_1 \phi_2' = 0$$

$$\frac{d}{dx}\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_1' \phi_2 - \phi_1 \phi_2'}{\phi_2^2} = \frac{0}{\phi_2^2} = 0 \quad \text{since } \phi_2^2 \neq 0$$

$$\frac{d}{dx}\left(\frac{\phi_1}{\phi_2}\right) = 0 \Rightarrow \frac{\phi_1}{\phi_2} = \underset{\neq 0}{C} = \text{const} \Rightarrow \phi_1 = C \phi_2$$

$\therefore$ e'functions are unique up to a nonzero multiplicative constant. ∎

<u>Non-uniqueness of eigenfunctions with periodic BCs</u>

Result $p[\phi_2 \phi_1' - \phi_1 \phi_2'] = \text{const}$ is still valid, but this constant does not have to be zero.

$\therefore$ 1 e'value may have <u>more than one</u> associated e'functions.

<u>Ex</u> $\phi'' + \lambda \phi = 0$

$\phi(-L) = \phi(L)$ $\qquad \phi'(-L) = \phi'(L)$

<u>e'values:</u> $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$

<u>e'functions:</u> $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$

and

<u>e'value:</u> $\lambda = 0$

<u>e'function:</u> const or 1 (unique e'function)

<u>Note</u> We showed that e'functions that correspond to distinct e'values are orthogonal. What if two e'functions correspond to the same e'value? In this case, these e'functions will be at least linearly independent. A system of linearly independent functions can be transformed into a system of linearly independent mutually orthogonal functions by <u>Gram - Schmidt orthogonalization method</u>. Thus, it is possible to get an orthogonal set of e'functions that correspond to the e'value.

Adjoint operator

Let $\mathcal{L}$ be some differential operator. If we can write by repeatedely applying integration by parts:

$$\int_a^b v\,\mathcal{L}(u)\,dx = \int_a^b u\,\mathcal{L}^*(v)\,dx - H(x)\Big|_a^b \qquad (2)$$

then $\mathcal{L}^*$ is called an <u>adjoint operator</u> (to $\mathcal{L}$).

equation (2) can be written as

$$\int_a^b \left[u\,\mathcal{L}^*(v) - v\,\mathcal{L}(u)\right] dx = H(x)\Big|_a^b$$

An operator $\mathcal{L}$ is <u>self-adjoint</u> when

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad H(x)\Big|_a^b = 0$$

Ex $\mathcal{L} = \frac{d}{dx}$. What is $\mathcal{L}^*$? What is $H(x)$?

$$\int_a^b v\,\mathcal{L}(u)\,dx = \int_a^b v \frac{du}{dx}\,dx \underset{\text{by parts}}{\overset{\text{integration}}{=}}$$

$$\left(\begin{array}{ll} U = v & dV = \frac{du}{dx}dx \\ dU = \frac{dv}{dx}dx & V = u \end{array} \right)$$

$$= \left[u(x)v(x)\right]\Big|_a^b - \int_a^b u \frac{dv}{dx}\,dx$$

$$\therefore \int_a^b v \underbrace{\mathcal{L}(u)}_{\frac{d}{dx}u} \, dx = \left[u(x)v(x)\right]\Big|_a^b - \int_a^b u \frac{dv}{dx} \, dx$$

Hence, here $\mathcal{L}^* = -\frac{d}{dx}$, $H(x) = -u(x)v(x)$