University of Idaho

Lecture 31

<u>Claim</u> <u>Uniqueness of eigenfunctions</u> (<u>regular S.-L. problem</u>)

Eigenfunctions of the regular Sturm-Liouville problem are defined up to an arbitrary non-zero constant.

□ $\nearrow$ Assume that there are two different e'functions ϕ_1 and ϕ_2 that correspond to the same e'value λ.

$$\mathcal{L}(\phi_1) + \lambda\sigma\phi_1 = 0 \quad / \cdot \phi_2$$

$$\mathcal{L}(\phi_2) + \lambda\sigma\phi_2 = 0 \quad / \cdot \phi_1$$

$$\phi_2 \mathcal{L}(\phi_1) + \lambda\sigma \cancel{\phi_1} \phi_2 - \mathcal{L}(\phi_2) \cdot \phi_1 - \lambda\sigma \cancel{\phi_2} \phi_1 = 0$$

$$\therefore \quad \phi_2 \mathcal{L}(\phi_1) - \phi_1 \mathcal{L}(\phi_2) = 0$$

apply Lagrange identity:

$$0 = \int_a^b \Big[\underbrace{\phi_2 \mathcal{L}(\phi_1) - \phi_1 \mathcal{L}(\phi_2)}_{=0} \Big] dx = \frac{d}{dx}\left[p\left(\phi_2 \phi_1' - \phi_1 \phi_2'\right)\right]$$

$$\therefore \quad \frac{d}{dx}\left[p\left(\phi_2 \phi_1' - \phi_1 \phi_2'\right)\right] = 0$$

$$\therefore \quad p\left(\phi_2 \phi_1' - \phi_1 \phi_2'\right) = \text{const} \qquad \left(= [\;]\Big|_{x=a} \overset{\sigma}{=} [\,..\,]\Big|_{x=b} \right)$$

University of Idaho

Use BCs at $x=a$ to find this const

$$\beta_1 \phi_1(a) + \beta_2 \phi_1'(a) = 0 \qquad \beta_1 \phi_2(a) + \beta_2 \phi_2'(a) = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\phi_1'(a) = -\frac{\beta_1}{\beta_2}\phi_1(a) \qquad \phi_2'(a) = -\frac{\beta_1}{\beta_2}\phi_2(a)$$

$$p(\phi_2 \phi_1' - \phi_1 \phi_2') = \text{const} = p(a)\Big[\phi_2(a) \underbrace{\phi_1'(a)}_{-\frac{\beta_1}{\beta_2}\phi_1(a)} - \phi_1(a) \underbrace{\phi_2'(a)}_{-\frac{\beta_1}{\beta_2}\phi_2(a)}\Big] =$$

$$= p(a)\left(-\frac{\beta_1}{\beta_2}\right)\left[\cancel{\phi_2(a)\phi_1(a)} - \cancel{\phi_1(a)\phi_2(a)}\right] =$$

$$= 0$$

$$\therefore \quad \underbrace{p(x)}_{>0}\left[\phi_2(x)\phi_1'(x) - \phi_1(x)\phi_2'(x)\right] = 0 \quad \text{for all } x \in [a,b]$$

University of Idaho

$p(x) > 0 \quad \Rightarrow \quad \phi_2(x)\,\phi_1'(x) - \phi_1(x)\,\phi_2'(x) = 0$

$\phi_2 \phi_1' - \phi_1 \phi_2' = 0$

$$\frac{d}{dx}\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_1'\phi_2 - \phi_1\phi_2'}{\phi_2^2} = \frac{0}{\phi_2^2} = 0$$

since $\phi_2^2 > 0$ (ϕ_2 is not trivial, i.e. $\phi_2 \not\equiv 0$)

$\therefore \quad \frac{\phi_1}{\phi_2} = \underset{\neq 0}{\text{const}} \quad \Rightarrow \quad \phi_1 = \text{const} \cdot \phi_2$

Hence, if there are two e'functions associated w/ the same e'value, they are constant multiple of each other: $\phi_1 = C\phi_2$, $C \neq 0$. Therefore,

e'functions are unique up to a multiplicative constant.

<u>Non-uniqueness of e'functions w/ periodic BCs.</u>

Result $p[\phi_2\phi_1' - \phi_1\phi_2'] = \text{const}$ is still valid, but this constant does not have to be zero.

$\therefore$ 1 e'value may have <u>more than one</u> associated e'functions.

<u>Ex</u>
$$\phi'' + \lambda\phi = 0$$
$$\phi(-L) = \phi(L), \quad \phi'(-L) = \phi'(L)$$

<u>E'values:</u> $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots \qquad (\lambda_n > 0)$

<u>E'functions:</u> $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$

and

~~[illegible]~~

<u>e'value</u> : $\lambda_0 = 0$

<u>e'function</u> : 1 or $\text{const} \neq 0$ (unique e'function)

<u>Note</u> We showed that e'functions that correspond to distinct e'values are orthogonal. What if two e'functions correspond to the same e'value? In this case, they will be at least linearly independent. A system of linearly independent functions can be transformed in a system of mutually orthogonal linearly independent functions by

Gram-Schmidt orthogonalization method. Thus, it is possible to get an orthogonal set of e'functions that correspond to the same e'value.

Adjoint Operator

Let $\mathcal{L}$ be some differential operator. If we write by repeatedly applying integration by parts:

$$\int_a^b v\,\mathcal{L}(u)\,dx = \int_a^b u\,\mathcal{L}^*(v)\,dx - H(x)\Big|_a^b \quad (1)$$

then operator $\mathcal{L}^*$ is called adjoint operator to $\mathcal{L}$.

Equation (1) may be written

$$\int_a^b \left[u\,\mathcal{L}^*(v) - v\,\mathcal{L}(u) \right] dx = H(x)\Big|_a^b$$

An operator $\mathcal{L}$ is <u>self-adjoint</u> when

$$\mathcal{L} = \mathcal{L}^* \quad \text{and} \quad H(x)\Big|_a^b = 0$$

<u>Ex</u> $\mathcal{L} = \frac{d}{dx}$. What is $\mathcal{L}^*$? What is $H(x)$?

$$\int_a^b v\,\mathcal{L}(u)\,dx = \int_a^b \underbrace{v}_{U} \underbrace{\frac{du}{dx}\,dx}_{dV} \quad \underline{\text{by parts}} \quad \left| \begin{array}{ll} U = v & dV = \frac{du}{dx}\,dx \\ dU = \frac{dv}{dx}\,dx & V = u \end{array} \right|$$

$$= \Big[u(x)v(x) \Big]_{x=a}^{x=b} - \int_a^b u\,\frac{dv}{dx}\,dx$$

$$\therefore \int_a^b v\,\mathcal{L}(u)\,dx = \int_a^b v \frac{du}{dx}\,dx = -\int_a^b u \frac{dv}{dx}\,dx + \Big[u(x)\,v(x)\Big]\Big|_a^b$$

Hence, here $\mathcal{L}^* = -\frac{d}{dx}$ $\quad H(x) = -u(x)\,v(x)$

Operators and Matrix Theory

Recall Green's formula:

$$\int_a^b \left[u\,\mathcal{L}(v) - v\,\mathcal{L}(u)\right]dx = \left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]_a^b$$

$\mathcal{L}$ is self-adjoint if $\displaystyle\int_a^b \left[u\,\mathcal{L}(v) - v\,\mathcal{L}(u)\right]dx = 0$

University of Idaho

Matrix theory equivalency : consider two vectors $\vec{u}$ and $\vec{v}$

$$\vec{u} \cdot (A\vec{v}) = \sum_i u_i \left(\sum_j a_{ij} v_j \right) = \sum_i \sum_j a_{ij} u_i v_j$$

$A = (a_{ij})$: $n \times n$ matrix

$\vec{u}, \vec{v}$: n-vectors $n \times 1$

$$\vec{v} \cdot (B\vec{u}) = \sum_j v_j \left(\sum_i b_{ji} u_i \right) = \sum_j \sum_i b_{ji} v_j u_i$$

$$= \sum_j \sum_i b_{ji} u_i v_j$$

$B = (b_{ji})$: $n \times n$ matrix