Lecture 32

# Operators and Matrix Theory

Recall Green's formula:

$$\int_a^b \left[u\,\mathcal{L}(v) - v\,\mathcal{L}(u)\right] dx = \left[p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]\Bigg|_a^b$$

Self-adjoint operator $\Rightarrow$ $\boxed{\int_a^b \left[u\,\mathcal{L}(v) - v\,\mathcal{L}(u)\right] dx = 0}$

Matrix theory equivalent: consider two vectors $\vec{u}$ and $\vec{v}$

$$\vec{u}\cdot(A\vec{v}) = \sum_i u_i \left(\sum_j a_{ij} v_j\right) = \sum_i \sum_j a_{ij} u_i v_j$$

$A$: $n\times n$ matrix $\quad A = (a_{ij})$

$\vec{u}, \vec{v}$: $n$-vectors

$$\vec{v}\cdot(B\vec{u}) = \sum_j v_j \left(\sum_i b_{ji} u_i\right) = \sum_i \sum_j b_{ji} u_i v_j$$

B: $n \times n$ matrix

Let $B = A^T$ (transpose of A) $\Rightarrow$ $b_{ji} = a_{ij}$

$\therefore$ $\boxed{\vec{u} \cdot (A\vec{v}) - \vec{v} \cdot (A^T \vec{u}) = 0}$ analogue of Green's formula

$A^T$ is a transpose of matrix A $\Leftrightarrow$ $\mathcal{L}^*$ is an adjoint operator to operator $\mathcal{L}$

A is symmetric i.e. $A = A^T$ $\Leftrightarrow$ $\mathcal{L} = \mathcal{L}^*$ together with BCs i.e. $\mathcal{L}$ is self-adjoint

$\vec{u} \cdot (A\vec{v}) - \vec{v} \cdot (A\vec{u}) = 0$ $\Leftrightarrow$ Green's formula for self-adjoint operators

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## Orthogonality

We showed for self-adjoint operators that if e'values are distinct, then the corresponding e'functions are orthogonal.

Let $A$ be a symmetric matrix : $A = A^T$

Let $\vec{u}$ be an e'vector w/ associated e'value $\lambda_1$:

$$A\vec{u} = \lambda_1 \vec{u}$$

Let $\vec{v}$ be an e'vector w/ associated e'value $\lambda_2$:

$$A\vec{v} = \lambda_2 \vec{v}$$

We showed earlier that for symmetric matrices we have

$$\vec{u}\cdot(\underbrace{A\vec{v}}_{=\lambda_2\vec{v}}) - \vec{v}\cdot(\underbrace{A\vec{u}}_{=\lambda_1\vec{u}}) = 0 \quad \Rightarrow \quad \lambda_2 \vec{u}\cdot\vec{v} - \lambda_1 \vec{v}\cdot\vec{u} = 0$$

$(\lambda_2 - \lambda_1)\, \vec{u} \cdot \vec{v} = 0$

Since $\lambda_1 \neq \lambda_2 \Rightarrow \lambda_2 - \lambda_1 \neq 0 \Rightarrow \vec{u} \cdot \vec{v} = 0 \Rightarrow \vec{u} \perp \vec{v}$

Hence, e'vectors that correspond to distinct e'values are orthogonal.

## Real e'values

For S.-L. problem, we showed that e'values are real.

Let $A$ be a real symmetric matrix.

Let $\vec{u}$ be e'vector of $A$ w/ associated e'value $\lambda$.

$$A\vec{u} = \lambda \vec{u}$$

$$\overline{A\vec{u}} = \overline{\lambda \vec{u}} \Rightarrow \underset{\text{real}}{A}\, \overline{\vec{u}} = \bar{\lambda} \cdot \overline{\vec{u}}$$

$\Rightarrow \bar{\lambda}$ is an e'value of $A$ w/ associated e'vector $\overline{\vec{u}}$

Recall

(1) $\vec{u}\,(A\vec{v}) - \vec{v}\cdot(A\vec{u}) = 0$ for symmetric $A$

In (1), let us use $\vec{u}$ as $\vec{u}$, $\vec{v} = \bar{\vec{u}}$.

$$\vec{u}\,\underbrace{(A\bar{\vec{u}})}_{=\bar{\lambda}\bar{\vec{u}}} - \bar{\vec{u}}\,\underbrace{(A\vec{u})}_{=\lambda\vec{u}} = 0$$

$$\Rightarrow (\bar{\lambda} - \lambda)\,\vec{u}\cdot\bar{\vec{u}} = 0$$

$\vec{u}\cdot\bar{\vec{u}} = |\vec{u}|^2 > 0$ since $\vec{u} \not\equiv \vec{0}$ as an e'vector

$\therefore\ \bar{\lambda} - \lambda = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ is real $\blacksquare$

## Completeness

<u>Recall</u>: any piecewise smooth function can be written as a superposition of Sturm-Liouville e'functions:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$ generalized Fourier series

<u>Matrix theory</u>: for any $n \times n$ real symmetric matrix it is always possible to construct $n$ linearly <u>independent</u> and <u>orthogonal</u> e'vectors $\vec{\phi}_1, .., \vec{\phi}_n$.

Then any vector $\vec{v} \in R^n$ can be written as a linear combination of these basis vectors $\vec{\phi}_i$, $i = 1, .., n$:

$$\vec{v} = \sum_{i=1}^{n} c_i \vec{\phi}_i \quad / \; \vec{\phi}_m$$

$$\vec{v} \cdot \vec{\phi}_m = \sum_{i=1}^{n} c_i \underbrace{\vec{\phi}_i \cdot \vec{\phi}_m}_{=0 \text{ if } i \neq m} = c_m \vec{\phi}_m \cdot \vec{\phi}_m$$

$$\therefore \quad \boxed{c_m = \frac{\vec{v} \cdot \vec{\phi}_m}{\vec{\phi}_m \cdot \vec{\phi}_m}}$$

$$\vec{\phi}_m \cdot \vec{\phi}_m = |\vec{\phi}_m|^2 > 0$$ since $\vec{\phi}_m \neq \vec{0}$ as e'vector

# Rayleigh Quotient

Recall regular Sturm-Liouville equation:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0 \qquad a \le x \le b$$

multiply both sides by $\phi$ and $\int_a^b \cdots$

$$\int_a^b \left[\phi\left(\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)\right) + q\phi^2\right]dx + \lambda\underbrace{\int_a^b \underbrace{\sigma}_{>0}\underbrace{\phi^2}_{\ge 0} dx}_{>0} = 0$$

$$\therefore \quad \lambda = \frac{-\int_a^b \left[\phi\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q\phi^2\right]dx}{\int_a^b \sigma\phi^2\,dx}$$

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$$\int_a^b \phi \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)dx = \left| \begin{matrix} U = \phi & dV = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) \\ dU = \phi' dx & V = p\phi' \end{matrix} \right| =$$

$$= \left[p(x)\phi(x)\phi'(x)\right]\Big|_a^b - \int_a^b p(\phi')^2 dx$$

$$\therefore \boxed{\lambda = \frac{-p\phi\phi'\Big|_a^b + \int_a^b \left[p(\phi')^2 - q\phi^2\right]dx}{\int_a^b \sigma\phi^2 dx} \equiv RQ[\phi]}$$

<u>Rayleigh Quotient</u>

or

$$\lambda = RQ[\phi]$$

## Minimization Principle

$$\lambda_1 = \min_u \frac{-puu'\big|_a^b + \int_a^b \left[p(u')^2 - qu^2\right]dx}{\int_a^b \sigma u^2\,dx}$$

where min is taken on all possible continuous functions that satisfy the same BCs as e'function $\phi$.

Let $u_T$: trial function, an approximation to e'function $\phi(x)$

$$\lambda_1 \le \frac{-pu_T u_T'\big|_a^b + \int_a^b \left[p(u_T')^2 - q(u_T)^2\right]dx}{\int_a^b \sigma u_T^2\,dx} = RQ[u_T]$$