Lecture 32

Let $B = A^T$ (transpose of A) $\Rightarrow$ $b_{ji} = a_{ij}$

$\therefore$ $\boxed{\vec{u}\cdot(A\vec{v}) - \vec{v}\cdot(A\vec{u}) = 0}$ <u>analogue of Green's formula</u>

A^T is a transpose of A $\Leftrightarrow$ $\mathcal{L}^*$ is an adjoint operator to operator $\mathcal{L}$

A is symmetric i.e. $A^T = A$ $\Leftrightarrow$ $\mathcal{L} = \mathcal{L}^*$ together with BCs i.e. $\mathcal{L}$ is self-adjoint

$\vec{u}\cdot(A\vec{v}) - \vec{v}\cdot(A\vec{u}) = 0$ $\Leftrightarrow$ Green's formula for self-adjoint operators

$$\int_a^b [u\,\mathcal{L}(v) - v\,\mathcal{L}(u)]\,dx = 0$$

Orthogonality

We showed for self-adjoint operators that if e'values are distinct, then their corresponding e'functions associated with these e'values are orthogonal.

Let A be a symmetric matrix: $A^T = A$

Let $\vec{u}$ be an e'vector of A associated w/ e'value λ_1:

$$A\vec{u} = \lambda_1 \vec{u}$$

Let $\vec{v}$ be an e'vector of A associated w/ e'value λ_2:

$$A\vec{v} = \lambda_2 \vec{v}$$

For symmetric matrices we have

$$\vec{u}\cdot(\underbrace{A\vec{v}}_{=\lambda_2\vec{v}}) - \vec{v}\cdot(\underbrace{A\vec{u}}_{=\lambda_1\vec{u}}) = 0$$

dot product $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

$$\lambda_2\,\vec{u}\cdot\vec{v} - \lambda_1\,\vec{v}\cdot\vec{u} = 0$$

$$\underbrace{(\lambda_2-\lambda_1)}_{\neq 0 \text{ since } \lambda_1,\lambda_2 \text{ are distinct}}\vec{u}\cdot\vec{v} = 0 \quad\Rightarrow\quad \vec{u}\cdot\vec{v}=0 \quad\Rightarrow\quad \vec{u}\perp\vec{v}$$

Hence, e'vectors that correspond to distinct e'values are orthogonal. ■

Real e'values

For S.-L. problem, we showed that e'values are real.

Let A be a real symmetric matrix.

Let $\vec{u}$ be an e'vector of A w/ ~~assoc~~ associated e'value λ.

$$A\vec{u} = \lambda \vec{u}$$

Take complex conjugate of both sides of eqⁿ:

$$\overline{A\vec{u}} = \overline{\lambda \vec{u}} \Rightarrow \underbrace{\bar{A}}_{=A} \cdot \bar{\vec{u}} = \bar{\lambda} \cdot \bar{\vec{u}} \Rightarrow A \cdot \bar{\vec{u}} = \bar{\lambda} \cdot \bar{\vec{u}}$$

$\therefore \bar{\lambda}$ is an e'value of A w/ e'vector $\bar{\vec{u}}$.

Let's use an analogue of Green's formula

$$\vec{u} \cdot (A\vec{v}) - \vec{v} \cdot (A\vec{u}) = 0 \qquad \text{with } \vec{u} = \vec{u},\ \vec{v} = \bar{\vec{u}}$$

i.e. $\vec{u} \cdot \underbrace{(A\bar{\vec{u}})}_{=\bar{\lambda} \cdot \bar{\vec{u}}} - \bar{\vec{u}} \cdot \underbrace{(A\vec{u})}_{\lambda \cdot \vec{u}} = 0$

$$\bar{\lambda}\cdot\vec{u}\cdot\bar{\vec{u}} - \lambda\cdot\bar{\vec{u}}\cdot\vec{u} = 0$$

$$(\bar{\lambda}-\lambda)\cdot\underbrace{(\vec{u}\cdot\bar{\vec{u}})}_{\substack{|\vec{u}|^2 \\ >0}} = 0$$

$\vec{u} = u_r + i\,u_i$

$\bar{\vec{u}} = u_r - i\cdot u_i$

$\vec{u}\cdot\bar{\vec{u}} = u_r^2 + u_i^2 = |\vec{u}|^2 > 0$

since $\vec{u}$ is e'vector, i.e.

$\vec{u} \not\equiv \vec{0}$

$\therefore\ \bar{\lambda}-\lambda = 0$ or $\lambda = \bar{\lambda}$

i.e. λ is a real #.

Completeness

Recall that any piecewise smooth function can be written as a generalized Fourier series or linear superposition of S.-L. e'functions:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$: generalized Fourier series

<u>Matrix theory</u>: for any $n \times n$ real symmetric matrix it is always possible to construct n linearly independent and orthogonal e'vectors $\vec{\phi}_1, \dots, \vec{\phi}_n$. Then any vector $\vec{v} \in \mathbb{R}^n$ can be written as a linear combination of these basis vectors $\vec{\phi}_i$, $i = 1, \dots, n$:

$$\vec{v} = \sum_{i=1}^{n} c_i \vec{\phi}_i \qquad / \cdot \vec{\phi}_m$$

$$\vec{v} \cdot \vec{\phi}_m = \sum_{i=1}^{n} c_i \underbrace{\vec{\phi}_i \cdot \vec{\phi}_m}_{=0}$$ if $i \neq m$ and $\neq 0$ if $i = m$

$$\boxed{c_m = \frac{\vec{v}\cdot\vec{\phi}_m}{\vec{\phi}_m\cdot\vec{\phi}_m}}$$

$\vec{\phi}_m \cdot \vec{\phi}_m = |\vec{\phi}_m|^2 > 0$ since $\vec{\phi}_m \neq \vec{0}$ as an e'vector

Rayleigh Quotient

Recall the regular Sturm-Liouville equation:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0 \qquad a \le x \le b$$

multiply both sides by ϕ and $\int_a^b \ldots$

$$\int_a^b \left[\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)\phi + q\phi^2\right]dx + \lambda \underbrace{\int_a^b \underbrace{\sigma}_{>0}\underbrace{\phi^2}_{\ge 0}dx}_{>0} = 0$$

$$\therefore \quad \lambda = \frac{-\int_a^b \left[\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)\phi + q\phi^2 \right] dx}{\int_a^b \phi^2 \sigma \, dx}$$

$$\int_a^b \underbrace{\phi}_{U} \underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) dx}_{dV} \overset{\text{by}}{\underset{\text{parts}}{=}} \left| \begin{array}{ll} U = \phi & dV = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) \\ dU = \frac{d\phi}{dx} dx & V = p(x)\frac{d\phi}{dx} \end{array} \right| =$$

$$= \phi(x)\frac{d\phi}{dx} p(x) \Big|_a^b - \int_a^b p(x)\left(\frac{d\phi}{dx}\right)^2 dx$$

$$\frac{d}{dx} = \, '$$

then

$$\boxed{\lambda = \frac{-p\phi\phi'\big|_a^b + \int_a^b \left[p(\phi')^2 - q\phi^2\right]dx}{\int_a^b \phi^2 \sigma\, dx} = RQ[\phi]}$$

<u>Rayleigh Quotient</u>

i.e. $\lambda = RQ[\phi]$

<u>Minimization Principle</u>

$$\lambda_1 = \min_u \frac{-puu'\big|_a^b + \int_a^b \left[p(u')^2 - qu^2\right]dx}{\int_a^b u^2 \sigma\, dx}$$

where min is taken over all possible continuous functions that satisfy the same BCs as efunction ϕ_1.

Let u_T : trial function, an approximation to e'function ϕ_1. Then

$$\lambda_1 \leq \frac{-p u_T u_T' \Big|_a^b + \int_a^b \left[p (u_T')^2 - q u_T^2 \right] dx}{\int_a^b u_T^2 \cdot \sigma \, dx} = RQ[u_T]$$

Ex Consider

$$\phi'' + \lambda \phi = 0 \qquad \phi(0) = \phi(1) = 0 \qquad L = 1$$

$$\lambda_n = \left(\frac{n\pi}{1}\right)^2 = (n\pi)^2, \quad n = 1, 2 \ldots$$

$$\phi_n = \sin \frac{n\pi x}{1} = \sin n\pi x$$

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$\lambda_1 = \pi^2 \approx 9.8696$

$\phi_1(x) = \sin \pi x$