Lecture 33

Rayleigh Quotient (Cont'd)

Ex Consider

$$\phi'' + \lambda \phi = 0 \qquad \phi(0) = \phi(1) = 0 \qquad L = 1$$

We know that

$$\lambda_n = \left(\frac{n\pi}{1}\right)^2 = (n\pi)^2, \quad n = 1, 2, \ldots$$

$$\phi_n(x) = \sin \frac{n\pi x}{1} = \sin(n\pi x)$$

$$\lambda_1 = \pi^2 \approx \boxed{9.8696}$$

$$\phi_1(x) = \sin \pi x$$

minimization Principle:

$$\lambda_1 \leq \frac{-p u_T u_T' \Big|_0^1 + \int_0^1 \left[p (u_T')^2 - \cancel{q} (u_T)^2 \right] dx}{\int_0^1 \sigma u_T^2 \, dx}$$

u_T: trial function

$u_T(0) = u_T(1) = 0$

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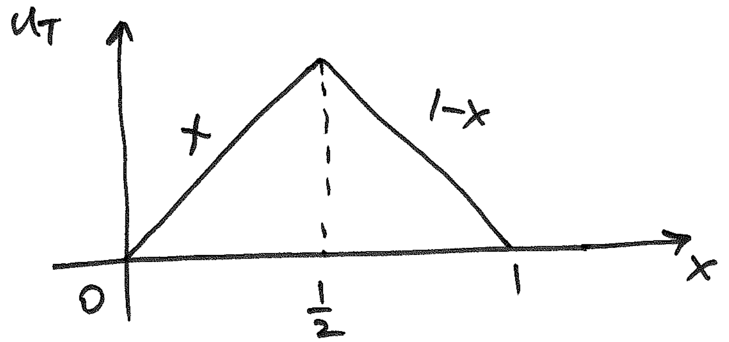
Here $p=1$, $q=0$, $\sigma=1$

$$u_T(0)=u_T(1)=0 \Rightarrow -\overset{1}{p}u_T u_T'\Big|_0^1 = -u_T u_T'\Big|_0^1 = -\Big(\overset{0}{u_T(1)}u_T'(1) - \underset{0}{u_T(0)}u_T'(0)\Big) = 0$$

$$\therefore \quad \lambda_1 \le \frac{\int_0^1 (u_T')^2\,dx}{\int_0^1 u_T^2\,dx} \qquad (*)$$

We know from Sturm-Liouville theorem that $\phi_1(x)$ has no roots in $(0,1)$. We also know that $u_T(0)=u_T(1)=0$, the same BCs as for $\phi_1(x)$.

(a)



Let

$$u_T(x)=\begin{cases} x, & 0\le x\le \frac{1}{2} \\ 1-x, & \frac{1}{2}\le x\le 1 \end{cases}$$

$$\lambda_1 \le \frac{\int_0^1 (u_T')^2\,dx}{\int_0^1 u_T^2\,dx} = \frac{\int_0^{1/2} 1^2\,dx + \int_{1/2}^1 (-1)^2\,dx}{\int_0^{1/2} x^2\,dx + \int_{\frac{1}{2}}^1 (1-x)^2\,dx} = \frac{1}{\frac{1}{24}+\frac{1}{24}}$$

$$= \boxed{12}$$

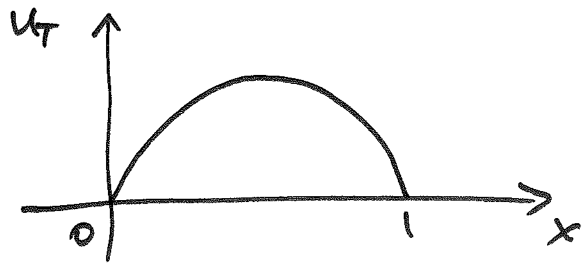
"close" to exact $\lambda_1 = 9.8696$

<u>Note</u> : u_T should satisfy the <u>same</u> BCs as $\phi_1(x)$:

$$u_T(0) = u_T(1) = 0$$

and have no roots on $(0,1)$, similarly to $\phi_1(x)$.

(b) $u_T(x) = x - x^2$



$$\lambda_1 \le \frac{\int_0^1 (1-2x)^2\,dx}{\int_0^1 (x-x^2)^2\,dx} =$$

$$= \frac{\int_0^1 (1-4x+4x^2)\,dx}{\int_0^1 (x^2-2x^3+x^4)\,dx} = \boxed{10}$$

closer to λ_1 than in case (a)

(Note: the choice $u_T = x - x^2$ gives a smooth trial function similar to $\varphi_1(x)$)

Proof of minimization Principle:

$$\lambda_1 \leq RQ[u]$$

where u is all possible continuous functions with the same BC as ϕ_1, and

$$\lambda_1 = RQ[\phi_1]$$

$$\mathcal{L}(u) = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)u \quad : \quad \text{Sturm-Liouville operator}$$

Sturm-Liouville eq^n can be written

$$\mathcal{L}(u) + \lambda\sigma u = 0 \quad / \, u$$

$$u\mathcal{L}(u) + \lambda\sigma u^2 = 0 \qquad \int_a^b$$

(1) $\therefore \quad \lambda = - \dfrac{\int_a^b u\, \mathcal{L}(u)\, dx}{\int_a^b u^2 \sigma\, dx} = RQ[u]$ same derivation as for RQ

E'functions $\{\phi_n\}$ form a complete set $\Rightarrow$ since u is an arbitrary continuous function, one can write

$$u = \sum_{n=1}^{\infty} a_n \phi_n(x) \quad \text{generalized Fourier series}$$

$$\mathcal{L}(u) = \mathcal{L}\left(\sum_{n=1}^{\infty} a_n \phi_n(x)\right) \underset{\substack{\text{linear}\\\text{operator}}}{\overset{\mathcal{L}\text{ is}}{=}} \sum_{n=1}^{\infty} a_n\, \mathcal{L}(\phi_n(x)) =$$

$\phi_n(x)$ satisfies S.-L. eq^n $\Rightarrow$ $\mathcal{L}(\phi_n) + \lambda_n \sigma \phi_n = 0$

$$\Rightarrow \mathcal{L}(\phi_n) = -\lambda_n \sigma \phi_n$$

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$$= \sum_{n=1}^{\infty} a_n (-\lambda_n) \sigma \phi_n = -\sum_{n=1}^{\infty} a_n \lambda_n \sigma \phi_n$$

$$RQ[u] \overset{(1)}{=} -\frac{\int_a^b u\, \mathcal{L}(u)\, dx}{\int_a^b u^2 \sigma\, dx} = \frac{\int_a^b \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n a_m \lambda_m \sigma \phi_n \phi_m\, dx}{\int_a^b \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n a_m \phi_n \phi_m \sigma\, dx} =$$

$$= \frac{\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n a_m \lambda_m \int_a^b \phi_n \phi_m \sigma\, dx}{\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n a_m \int_a^b \phi_n \phi_m \sigma\, dx} =$$

e'functions $\{\phi_n\}$ are orthogonal

$$= \frac{\sum_{n=1}^{\infty} a_n^2 \lambda_n \int_a^b \phi_n^2 \sigma\, dx}{\sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma\, dx} = RQ[u]$$

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But $\lambda_1 < \lambda_2 < \ldots$, hence

$$RQ[u] \geq \frac{\sum_{n=1}^{\infty} a_n^2 \lambda_1 \int_a^b \phi_n^2 \sigma \, dx}{\sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma \, dx} = \lambda_1$$

$$\therefore \quad \boxed{RQ[u] \geq \lambda_1}$$

If $u = \phi_1$, then $a_1 = 1$, $a_2 = a_3 = \ldots = 0$, then

$$RQ[\phi_1] = \frac{a_1^2 \lambda_1 \int_a^b \phi_1^2 \sigma \, dx}{a_1^2 \int_a^b \phi_1^2 \sigma \, dx} = \lambda_1 \quad \Rightarrow \quad \boxed{RQ[\phi_1] = \lambda_1}$$

∎

<u>Note</u> We can use the same approach to find λ_2. If we pick a function u that is orthogonal to ϕ_1 $\Rightarrow$

$$a_1 = \frac{\int_a^b u\,\phi_1\,\sigma\,dx}{\int_a^b \phi_1^2\,\sigma\,dx} = 0$$

$$\therefore\ RQ[u] = \frac{\sum_{n=2}^{\infty} a_n^2\,\lambda_n \int_a^b \phi_n^2\,\sigma\,dx}{\sum_{n=2}^{\infty} a_n^2 \int_a^b \phi_n^2\,\sigma\,dx} \geq \lambda_2$$

<u>Note</u> We can continue this process to find approximations to other e'values.

<u>Analogy with matrix theory</u>

A: $n\times n$ real symmetric matrix

A has n real e'values and a complete set of orthogonal e'vectors.

Let $\vec{x}$ be a vector in $\mathbb{R}^n$. Define

$$\rho(\vec{x}) \overset{\text{def}}{=} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T x}$$: Rayleigh Quotient

Let $\vec{x}$ be an e'vector w/ associated e'value λ:

$$A\vec{x} = \lambda\vec{x}, \quad \vec{x} \neq \vec{0}$$

$$\rho(\vec{x}) = \frac{\vec{x}^T A \vec{x}}{\vec{x}^T x} = \frac{\vec{x}^T \lambda \vec{x}}{\vec{x}^T x} = \frac{\lambda(\vec{x}^T \vec{x})}{\vec{x}^T \vec{x}} = \lambda$$

$\vec{x}$ is e'vector of A

$\neq 0$ since $\vec{x} \neq \vec{0}$ as an e'vector

if α is an approximation to λ. Then

$$A\vec{x} - \alpha\vec{x} \longrightarrow \min$$

Solve this using, say, least squares method to find $\vec{x}$ and α.