Lecture 33

Rayleigh Quotient (Cont'd)

Ex Consider

$\phi'' + \lambda \phi = 0$ $\qquad \phi(0) = \phi(1) = 0 \qquad L = 1 \qquad p = 1,\ q = 0,\ \sigma = 1$

We know that $\lambda_n = \left(\frac{n\pi}{1}\right)^2 = (n\pi)^2, \qquad n = 1, 2, \dots$

$$\phi_n(x) = \sin \frac{n\pi x}{1} = \sin n\pi x$$

$$\phi_1(x) = \sin \pi x$$

$\lambda_1 = \pi^2 \approx \boxed{9.8696}$

minimization principle:

$$\lambda_1 \le \frac{-\overset{=1}{p} u_T \cancel{u_T'} \Big|_0^1 + \int_0^1 \left[\overset{=1}{p} (u_T')^2 - \cancel{q} (u_T)^2 \right] dx}{\int_0^1 u_T^2 \underset{=1}{\sigma}\, dx}$$

Here: u_T : trial function

$$u_T(0) = u_T(1) = 0$$

We have $p=1,\ q=0,\ \sigma=1$

$$u_T(0) = u_T(1) = 0 \Rightarrow -p u_T u_T' \Big|_0^1 = 0$$

$$\therefore \quad \lambda_1 \le \frac{\int_0^1 (u_T')^2 dx}{\int_0^1 u_T^2 dx}$$

We know from Sturm-Liouville theorem that $\phi_1(x)$ has no roots in $(0,1)$. We also know that $u_T(0) = u_T(1)$, the same BCs as for $\phi_1(x)$.

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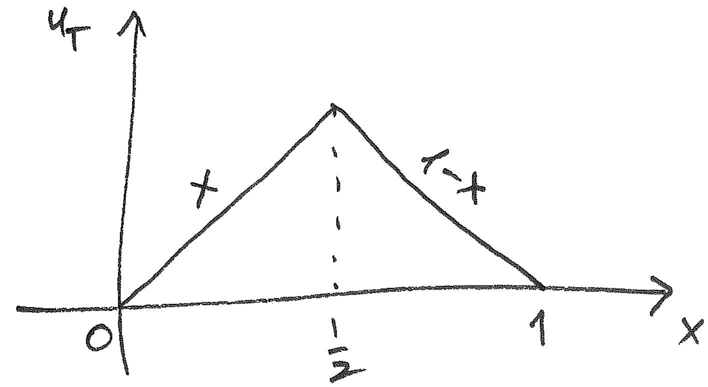
(a) Let

$$u_T(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases}$$



Then

$$\lambda_1 \le \frac{\int_0^1 (u_T')^2\,dx}{\int_0^1 u_T^2\,dx} = \frac{\int_0^{1/2} 1^2\,dx + \int_{1/2}^1 (-1)^2\,dx}{\int_0^{1/2} x^2\,dx + \int_{1/2}^1 (1-x)^2\,dx} = \frac{1}{\frac{1}{24}+\frac{1}{24}} = \boxed{12}$$

"close" to exact value 9.8696 (error is 21.59%)

$$u_T'(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{2} \\ -1, & \frac{1}{2} \le x \le 1 \end{cases}$$

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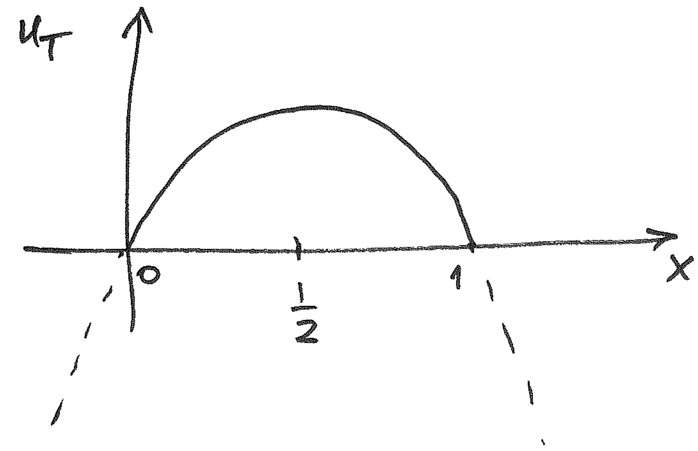
(b) $u_T(x) = x - x^2$



$u_T'(x) = 1 - 2x$

$$\lambda_1 \leq \frac{\int_0^1 (u_T')^2\,dx}{\int_0^1 u_T^2\,dx} = \frac{\int_0^1 (1-2x)^2\,dx}{\int_0^1 (x-x^2)^2\,dx} = \frac{\int_0^1 (1-4x+4x^2)\,dx}{\int_0^1 (x^2-2x^3+x^4)\,dx} = \boxed{10}$$

closer to the exact value $\lambda = 9.8696$

(error is 1.32%)

Note: in case (b), the trial function is also smooth as $\phi_1(x)$ and does not only have the same BCs & # of roots in $(0,1)$ as in case (a).

Proof of Minimization Principle.

$$\lambda_1 \leq RQ[u]$$

where u is all possible continuous functions w/ the same BCs as ϕ_1, and

$$\lambda_1 = RQ[\phi_1]$$

Recall

$$\mathcal{L}(u) = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u \; :$$ Sturm – Liouville operator

Sturm-Liouville equation can be written as

$$\mathcal{L}(u) + \lambda \sigma u = 0 \quad \Big| \cdot u$$

$$u\,\mathcal{L}(u) + \lambda \sigma u^2 = 0 \qquad \Big| \int_a^b$$

$$\therefore \quad (1) \quad \lambda = -\frac{\int_a^b u\,\mathcal{L}(u)\,dx}{\int_a^b u^2 \sigma\,dx} = RQ[u]$$

<u>Note</u>: the same derivation as for RQ

E'functions $\{\phi_n\}$ form a complete set. Since u is an arbitrary continuous function, we can write it as a generalized Fourier series

$$u = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

$$\mathcal{L}(u) = \mathcal{L}\Big(\sum_{n=1}^{\infty} a_n \phi_n(x)\Big) \underset{\text{a linear operator}}{\overset{\mathcal{L}\text{ is}}{=}} \sum_{n=1}^{\infty} a_n \mathcal{L}(\phi_n) =$$

$\phi_n(x)$ satisfies S.-L. equation $\Rightarrow$ $\mathcal{L}(\phi_n) + \lambda_n \sigma \phi_n = 0$

$$\therefore \quad \mathcal{L}(\phi_n) = -\lambda_n \sigma \phi_n$$

$$= \sum_{n=1}^{\infty} a_n (-\lambda_n \sigma) \phi_n = -\sum_{n=1}^{\infty} a_n \lambda_n \sigma \phi_n$$

$$RQ[u] \overset{(1)}{=} -\frac{\int_a^b u \mathcal{L}(u)\,dx}{\int_a^b u^2 \sigma\,dx} = \frac{\int_a^b \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \lambda_m \sigma \phi_n \phi_m\,dx}{\int_a^b \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \sigma \phi_n \phi_m\,dx} =$$

$$= \frac{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \lambda_m \int_a^b \phi_n \phi_m \sigma\,dx}{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n a_m \int_a^b \phi_n \phi_m \sigma\,dx} \quad \boxed{=}$$

ε' functions $\{\phi_n\}$ are mutually orthogonal

$$\Rightarrow \int_a^b \phi_n \phi_m \sigma \, dx = \begin{cases} 0, & n \neq m \\ \text{non-zero}, & n = m \end{cases}$$

$$\boxed{\equiv} \quad \frac{\sum_{n=1}^{\infty} a_n^2 \lambda_n \int_a^b \phi_n^2 \sigma \, dx}{\sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma \, dx} = RQ[u]$$

we know

$$\lambda_1 < \lambda_2 < \ldots$$

hence

$$RQ[u] \geq \frac{\sum_{n=1}^{\infty} a_n^2 \lambda_1 \int_a^b \phi_n^2 \sigma \, dx}{\sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma \, dx} = \lambda_1$$

$$\therefore \quad \boxed{\lambda_1 \leq RQ[u]}$$

If $u = \phi_1$, then $a_1 = 1$, $a_2 = a_3 = \ldots = 0$, and

$$RQ[\phi_1] = \frac{a_1^2 \lambda_1 \int_a^b \phi_1^2 \sigma \, dx}{a_1^2 \int_a^b \phi_1^2 \sigma \, dx} = \lambda_1$$

$$\therefore \quad \boxed{RQ[\phi_1] = \lambda_1} \quad \blacksquare$$

<u>Note</u> We can use the same approach to find λ_2. If we pick a function u that is orthogonal to ϕ_1 $\Rightarrow$

$$a_1 = \frac{\int_a^b u\,\phi_1\,\sigma\,dx}{\int_a^b \phi_1^2\,\sigma\,dx} = 0$$

$$\therefore \quad RQ[u] = \frac{\sum_{n=2}^{\infty} a_n^2 \lambda_n \int_a^b \phi_n^2\,\sigma\,dx}{\sum_{n=2}^{\infty} a_n^2 \int_a^b \phi_n^2\,\sigma\,dx} \geq \lambda_2$$

<u>Note</u> we can continue this process to find approximations to other eigenvalues.