Lecture 34

**Ex** Consider vibrations of a non-uniform string with constant tension $T_0$ but variable density $\rho(x)$ w/o sources ($Q=0$).

$$\rho u_{tt} = T_0 u_{xx} \qquad 0 < x < L$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x)$$

<u>Separation of variables</u> : $u(x,t) = \phi(x)\, h(t)$

<u>h-problem</u>

$$\frac{d^2 h}{dt^2} + \lambda h = 0 \;\Rightarrow\; h(t) = C_1 \cos\sqrt{\lambda}\, t + C_2 \sin\sqrt{\lambda}\, t$$

(assuming $\lambda > 0$
we can show this by using RQ )

## $\phi$-problem

$$T_0 \frac{d^2\phi}{dx^2} + \lambda \rho(x)\phi = 0$$

This is a regular S.-L. problem

$$\phi(0) = \phi(L) = 0$$

w/ $p = T_0$, $q = 0$, $\sigma = \rho(x)$

Rayleigh-Quotient:

$$\lambda = \frac{-\cancelto{0}{p\phi\phi'\Big|_0^L} + \int_0^L \left( p(\phi')^2 - \cancelto{0}{q\phi^2}\right) dx}{\int_0^L \phi^2 \sigma \, dx}$$

$$\lambda = \frac{\overset{>0}{T_0} \int_0^L (\phi')^2 \, dx}{\underbrace{\int_0^L \underset{\geq 0}{\phi^2} \underset{>0}{\rho} \, dx}_{>0}} \geq 0$$

$$\lambda = 0 \Rightarrow \int_0^L (\phi')^2 dx = 0 \Rightarrow \phi' \equiv 0 \Rightarrow \phi(x) \equiv \text{const}$$

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$\phi(0)=0 \Rightarrow \text{const}=0 \Rightarrow \phi\equiv 0$ : trivial solution ↯

$\therefore\ \lambda=0$ is not an e'value $\Rightarrow \boxed{\lambda>0}$

$$\therefore\ \boxed{u(x,t)=\sum_{n=1}^{\infty} a_n \sin\sqrt{\lambda_n}t\cdot\phi_n(x) + \sum_{n=1}^{\infty} b_n\cos\sqrt{\lambda_n}t\cdot\phi_n(x)}$$

ICs: $u(x,0)=f(x)=\sum_{n=1}^{\infty} b_n\,\phi_n(x) \quad \Big|\ \phi_m(x)\,\rho(x) \quad \int_0^L$

$$\therefore\ b_n=\frac{\int_0^L f(x)\,\phi_n(x)\,\rho(x)\,dx}{\int_0^L \phi_n^2(x)\,\rho(x)\,dx}$$

$$u_t(x,0)=g(x)=\sum_{n=1}^{\infty} \underbrace{a_n\sqrt{\lambda_n}}\cdot\phi_n(x)$$

$$a_n \sqrt{\lambda_n} = \frac{\int_0^L g(x)\,\phi_n(x)\,\rho(x)\,dx}{\int_0^L \phi_n^2(x)\,\rho(x)\,dx}$$

Let's find more information about e'values using Rayleigh Quotient.

$\sqrt{\lambda_1}$ : lowest frequency

Assume $\rho_{min} \le \rho \le \rho_{max}$ $\quad | \; u^2 \quad \int_0^L$

where $u(x)$ is a continuous function that satisfies the same BCs : $u(0) = u(L) = 0$ as e'functions $\phi_n(x)$.

$$\rho_{min} \int_0^L u^2(x)\,dx \le \int_0^L \rho(x)\,u^2(x)\,dx \le \rho_{max} \int_0^L u^2(x)\,dx$$

$$\frac{1}{\rho_{max}\int_0^L u^2(x)\,dx} \leq \frac{1}{\int_0^L u^2(x)\,\rho(x)\,dx} \leq \frac{1}{\rho_{min}\int_0^L u^2(x)\,dx} \qquad (1)$$

<u>Minimization Principle:</u>

$$\lambda_1 = \min_u RQ[u] = \min_u \frac{\cancel{-pu u'\Big|_0^L}^{\,0} + \int_0^L \left(p(u')^2 - \cancel{qu^2}^{\,0}\right)dx}{\int_0^L u^2 \sigma\,dx}$$

$u$ is cts

$u(0) = u(L) = 0$

<u>$\phi$-problem:</u>

$$T_0 \frac{d^2\phi}{dx^2} + \lambda \rho(x)\,\phi = 0$$

$$\phi(0) = \phi(L) = 0$$

$p = T_0$

$q = 0$

$\sigma = \rho$

$$\therefore \lambda_1 = \min_u \frac{\int_0^L T_0 (u')^2\,dx}{\int_0^L u^2 \rho(x)\,dx} = T_0 \min_u \frac{\int_0^L (u')^2\,dx}{\int_0^L u^2 \rho(x)\,dx}$$

$\rho(x)$ doesn't depend on $u$, so we could use (1) to write

$$\frac{T_0}{\rho_{max}} \min_u \frac{\int_0^L (u')^2\,dx}{\int_0^L u^2\,dx} \le \lambda_1 = T_0 \min_u \frac{\int_0^L (u')^2\,dx}{\int_0^L u^2 \rho(x)\,dx} \le \frac{T_0}{\rho_{min}} \min \frac{\int_0^L (u')^2\,dx}{\int_0^L u^2\,dx} \qquad (2)$$

But $\min_u \dfrac{\int_0^L (u')^2\,dx}{\int_0^L u^2\,dx}$ subject to BCs $u(0)=u(L)=0$

can be computed (as Rayleigh Quotient) from the following S.-L. problem w/ constant coefficients:

$\phi'' + \mu\phi = 0$ $\qquad p=1,\ q=0,\ \sigma=1$

$\phi(0)=\phi(L)=0$

$\therefore\ \mu_n = \left(\frac{n\pi}{L}\right)^2,\ n=1,2,\dots$ $\qquad \phi_n(x) = \sin\frac{n\pi x}{L},\ n=1,2,3,\dots$

and

$$\mu_1 = \min_{u} \frac{\int_0^L (u')^2\,dx}{\int_0^L u^2\,dx}$$

$$\parallel$$

$n=1$ $\quad \left(\frac{\pi}{L}\right)^2$

$\therefore$ inequalities in (2) become

$$\frac{T_0}{\rho_{max}} \mu_1 \le \lambda_1 \le \frac{T_0}{\rho_{min}} \cdot \mu_1$$

$$\frac{T_0}{\rho_{max}} \cdot \left(\frac{\pi}{L}\right)^2 \le \lambda_1 \le \frac{T_0}{\rho_{min}} \cdot \left(\frac{\pi}{L}\right)^2$$

Then the fundamental frequency $\sqrt{\lambda_1}$ is

$$\boxed{\sqrt{\frac{T_0}{\rho_{max}}} \cdot \frac{\pi}{L} \le \sqrt{\lambda_1} \le \sqrt{\frac{T_0}{\rho_{min}}} \cdot \frac{\pi}{L}}$$

lower and upper bounds for the lowest frequency $\sqrt{\lambda_1}$

Ex Boundary conditions of the $3^{rd}$ kind

$u_t = k\, u_{xx}$ or $u_{tt} = c^2 u_{xx}$ $\quad k, c$ : const

$u(0,t) = 0$, $\quad u_x(L,t) = -h\, u(L,t)$, $\quad h \neq 0$

Separation of variables: $u(x,t) = G(t)\,\phi(x)$

Time:

$$\frac{dG}{dt} + \lambda k\, G = 0$$ for heat eq$^{n}$

$$\frac{d^2G}{dt^2} + \lambda c^2 G = 0$$ for wave eq$^{n}$ / vibrating string

$\Phi$- problem (spatial))

This is a regular S.-L. problem w/ $p=1$, $q=0$, $\sigma=1$

$$\frac{d^2\Phi}{dx^2} + \lambda \Phi = 0$$

$\Phi(0)=0$ , $\quad \Phi'(L) = -h\,\Phi(L)$ $\quad$ or $\quad \Phi'(L) + h\,\Phi(L) = 0$

Rayleigh Quotient:

$$\lambda = \frac{-p\Phi\Phi'\Big|_0^L + \int_0^L \big[\underset{=1}{p}(\Phi')^2 - \cancelto{0}{q\Phi^2}\big]\,dx}{\int_0^L \Phi^2 \underset{=1}{\sigma}\,dx}$$

$$-\underset{=1}{p}\Phi\Phi'\Big|_0^L = -\Phi\Phi'\Big|_0^L = -\Phi(L)\underbrace{\Phi'(L)}_{=-h\Phi(L)} + \cancelto{0}{\Phi(0)}\Phi'(0) = h\,\Phi^2(L)$$

$$\therefore \lambda = \frac{h\,\overbrace{\Phi^2(L)}^{\geq 0} + \overbrace{\int_0^L (\Phi')^2\,dx}^{\geq 0}}{\int_0^L \Phi^2\,dx} \quad ,$$

$h \neq 0$ const

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1. If $h>0 \Rightarrow \lambda>0$
2. If $h<0 \Rightarrow \lambda$ is negative if $|h\,\phi^2(L)| > \int_0^L (\phi')^2 dx$

<u>Positive e'values:</u> $\lambda>0$

If $h>0 \Rightarrow \lambda>0$ (from RQ)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad \phi(0)=0 \qquad \phi'(L) + h\,\phi(L) = 0$$

$\phi(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$\phi(0)=0 \Rightarrow C_1 = 0 \Rightarrow \phi(x) = C_2 \sin\sqrt{\lambda}x$

$\phi'(x) = C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$\phi'(L) + h\phi(L) = 0 \Rightarrow C_2\sqrt{\lambda}\cos\sqrt{\lambda}L + h\,C_2\sin\sqrt{\lambda}L = 0$