Lecture 34

Analogy with matrix theory

A: $n \times n$ real symmetric matrix

A has n real e'values and a complete set of orthogonal vectors

Let $\vec{x}$ be a vector in R^n. Define

$$\rho(\vec{x}) \overset{def}{=} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} \quad : \quad \underline{\text{Rayleigh quotient}}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\vec{x}^T = (x_1 \dots x_n)$$

Let $\vec{x}$ be an e'vector of A with associated e'value λ: $\quad A\vec{x} = \lambda \vec{x}$

$$\rho(\vec{x}) = \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \frac{\vec{x}^T \lambda \vec{x}}{\vec{x}^T \vec{x}} = \lambda \frac{\vec{x}^T \vec{x}}{\vec{x}^T \vec{x}} = \lambda$$

$\vec{x}$: e'vector $\Rightarrow$ $\vec{x} \neq \vec{0}$ $\Rightarrow$ $\vec{x}^T \vec{x} = \|\vec{x}\|^2 > 0$, $\neq 0$

Hence $\rho(\vec{x}) = \lambda$ if $\lambda, \vec{x}$: e'value – e'vector pair

If α is an approximation of λ, then

$$A\vec{x} - \alpha\vec{x} \to \min$$

Solve this problem using, say, the least squares method to find $\vec{x}$ and λ (their approximations).

(Math 432 Numerical Linear Algebra
More on finding e'values & e'vectors)

__Ex__ Consider vibrations of a non-uniform string with constant tension T_0 but variable density $\rho(x)$ w/o sources ($Q=0$).

$$\rho u_{tt} = T_0 u_{xx} \qquad 0 < x < L$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

<u>Separation of variables:</u> $u(x,t) = \phi(x)\, h(t)$

<u>h-problem</u>

$$\frac{d^2 h}{dt^2} + \lambda h = 0 \Rightarrow h(t) = C_1 \cos\sqrt{\lambda}t + C_2 \sin\sqrt{\lambda}t$$

(assuming $\lambda > 0$

we can show this by using RQ)

ϕ- problem

$$T_0 \frac{d^2\phi}{dx^2} + \lambda \rho(x)\, \phi = 0$$

$$\phi(0) = \phi(L) = 0$$

This is a regular S.-L. problem with

$p = T_0, \quad q = 0, \quad \sigma = \rho(x)$

Rayleigh Quotient:

$$\lambda = \frac{\underbrace{-p\phi\phi'\Big|_0^L}_{\to 0} + \int_0^L \Big[\overset{T_0}{\overset{\shortparallel}{p}}(\phi')^2 - \underbrace{\cancel{q\phi^2}}_{\to 0}\Big]\,dx}{\int_0^L \phi^2 \underset{\rho}{\underset{\shortparallel}{\sigma}}\,dx}$$

$$\lambda = \frac{\overset{>0}{T_0} \int_0^L \overset{\geq 0}{(\phi')^2}\,dx}{\int_0^L \underset{\geq 0}{\phi^2} \underset{>0}{\rho}\,dx} \geq 0$$

>0

$\nearrow$ $\lambda = 0 \Rightarrow \int_0^L \underbrace{(\phi')^2}_{\geq 0} dx = 0 \Rightarrow \phi' = 0$

$$\left.\begin{array}{l} \Rightarrow \phi = \text{const} \\ \phi(0) = \phi(L) = 0 \end{array}\right\} \Rightarrow \phi \equiv 0$$

$\Downarrow$

$\lambda = 0$ is not an e'value

$\therefore$ $\boxed{\lambda > 0}$

λ_n : e'values, $\lambda_n > 0$

$\phi_n(x)$: associated e'functions

Then

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \sin\sqrt{\lambda_n}\, t \cdot \phi_n(x) + \sum_{n=1}^{\infty} b_n \cos\sqrt{\lambda_n}\, t \cdot \phi_n(x)}$$

ICs: $u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$

From orthogonality of $\{\phi_n(x)\}$:

$$b_n = \frac{\int_0^L f(x)\,\phi_n(x)\,\rho(x)\,dx}{\int_0^L \phi_n^2(x)\,\rho(x)\,dx}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \underbrace{a_n \cdot \sqrt{\lambda_n}} \cdot \phi_n(x) = g(x)$$

$$a_n \cdot \sqrt{\lambda_n} = \frac{\int_0^L g(x)\,\phi_n(x)\,\rho(x)\,dx}{\int_0^L \phi_n^2(x)\,\rho(x)\,dx}$$

Solve for a_n.

Let's find more information about e'values using Rayleigh quotient.

$\sqrt{\lambda_1}$: lowest frequency

Assume $\rho_{min} \leq \rho(x) \leq \rho_{max}$ $\quad / \, u^2 \quad \int_0^L$

Multiply both sides by u^2 and integrate $\int_0^L$.

Here $u(x)$ is a continuous function that satisfies the same BCs: $u(0)=u(L)=0$ as e'functions $\phi(x)$.

$$\rho_{min} \int_0^L u^2(x)dx \leq \int_0^L u^2(x)\rho(x)\,dx \leq \rho_{max} \int_0^L u^2(x)dx$$

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(1) $$\frac{1}{\rho_{max}\int_0^L u^2(x)\,dx} \leq \frac{1}{\int_0^L u^2(x)\rho(x)\,dx} \leq \frac{1}{\rho_{min}\int_0^L u^2(x)\,dx}$$

Minimization Principle:

$$\lambda_1 = \min_u RQ[u] = \min_u \frac{-\cancelto{0}{p u u'}\big|_0^L + \int_0^L \left(\overset{T_0}{p}(u')^2 - \cancelto{0}{q u^2}\right)dx}{\int_0^L u^2 \underset{\rho}{\sigma}\,dx}$$

u: CTS functions

$u(0)=u(L)=0$

$$\lambda_1 = \min_u \frac{T_0\int_0^L (u')^2\,dx}{\int_0^L u^2\cdot\rho\,dx}$$

$\rho(x)$ does not depend on u, so we could use (1) to write

$$\frac{T_0}{\rho_{max}} \min_u \frac{\int_0^L (u')^2 dx}{\int_0^L u^2 dx} \le \lambda_1 = T_0 \min \frac{\int_0^L (u')^2 dx}{\int_0^L u^2 \rho(x) dx} \le \frac{T_0}{\rho_{min}} \min \frac{\int_0^L (u')^2 dx}{\int_0^L u^2 dx} \quad (2)$$

But

$$\min_u \frac{\int_0^L (u')^2 dx}{\int_0^L u^2 dx}$$

subject to BCs $u(0) = u(L) = 0$

can be computed as Rayleigh quotient from the following problem:

$$\phi'' + \mu \phi = 0 \qquad p=1, \quad q=0, \quad \sigma = 1$$

$$\phi(0) = \phi(L) = 0$$

$$\therefore \quad \mu_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \ldots$$

$$\phi_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$$

and

$$\mu_1 = \min_u \frac{\int_0^L (u')^2\,dx}{\int_0^L u^2\,dx}$$

$$\mu_1 = \left(\frac{\pi}{L}\right)^2$$

Inequalities in (2) become

$$\frac{T_0}{\rho_{max}}\,\mu_1 \le \lambda_1 \le \frac{T_0}{\rho_{min}}\,\mu_1$$

$$\frac{T_0}{\rho_{max}}\cdot\left(\frac{\pi}{L}\right)^2 \le \lambda_1 \le \frac{T_0}{\rho_{min}}\cdot\left(\frac{\pi}{L}\right)^2$$

Then the fundamental frequency $\sqrt{\lambda_1}$ is

$$\boxed{\sqrt{\frac{T_0}{\rho_{max}}}\cdot\frac{\pi}{L} \le \sqrt{\lambda_1} \le \sqrt{\frac{T_0}{\rho_{min}}}\cdot\frac{\pi}{L}}$$

lower & upper bounds for the fundamental frequency $\sqrt{\lambda_1}$

<u>Ex</u> Boundary conditions of the 3^{rd} kind.

$u_t = k u_{xx}$ or $u_{tt} = c^2 u_{xx}$ $\quad k, c$: const

$u(0,t)=0$ $\quad u_x(L,t) = -h\,u(L,t)$ $\quad h \neq 0$

Separation of variables:

$$u(x,t) = G(t)\,\phi(x)$$

Time-problems:

$$\frac{dG}{dt} + \lambda k G = 0 \qquad \text{for heat eq}^{n}$$

$$\frac{d^2G}{dt^2} + \lambda c^2 G = 0 \qquad \text{for wave eq}^{n}$$

ϕ-problem (spatial problem)

This is a regular S.-L. problem w/ $p=1$, $q=0$, $\sigma=1$.

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

$$\phi(0)=0, \qquad \phi'(L) = -h\,\phi(L)$$

$$\text{or} \quad \phi'(L) + h\,\phi(L) = 0$$