Lecture 35

From last lecture:

$$C_2 \sqrt{\lambda} \cos \sqrt{\lambda} L + h C_2 \sin \sqrt{\lambda} L = 0$$

$$\underbrace{C_2}_{\neq 0} (\sqrt{\lambda} \cos \sqrt{\lambda} L + h \sin \sqrt{\lambda} L) = 0$$

$$\Rightarrow \sqrt{\lambda} \cos \sqrt{\lambda} L + h \sin \sqrt{\lambda} L = 0$$

or $\boxed{\tan \sqrt{\lambda} L = -\frac{1}{h} \sqrt{\lambda}}$ transcendental eq^{n}

Let $x = \sqrt{\lambda} L$. Denote $f_1(x) = \tan x$

$$-\frac{1}{h} \sqrt{\lambda} = -\frac{1}{hL} \underbrace{\sqrt{\lambda} L}_{x} = -\frac{1}{hL} x \equiv f_2(x) :$$ linear f^{n} w/ slope $-\frac{1}{hL} < 0$

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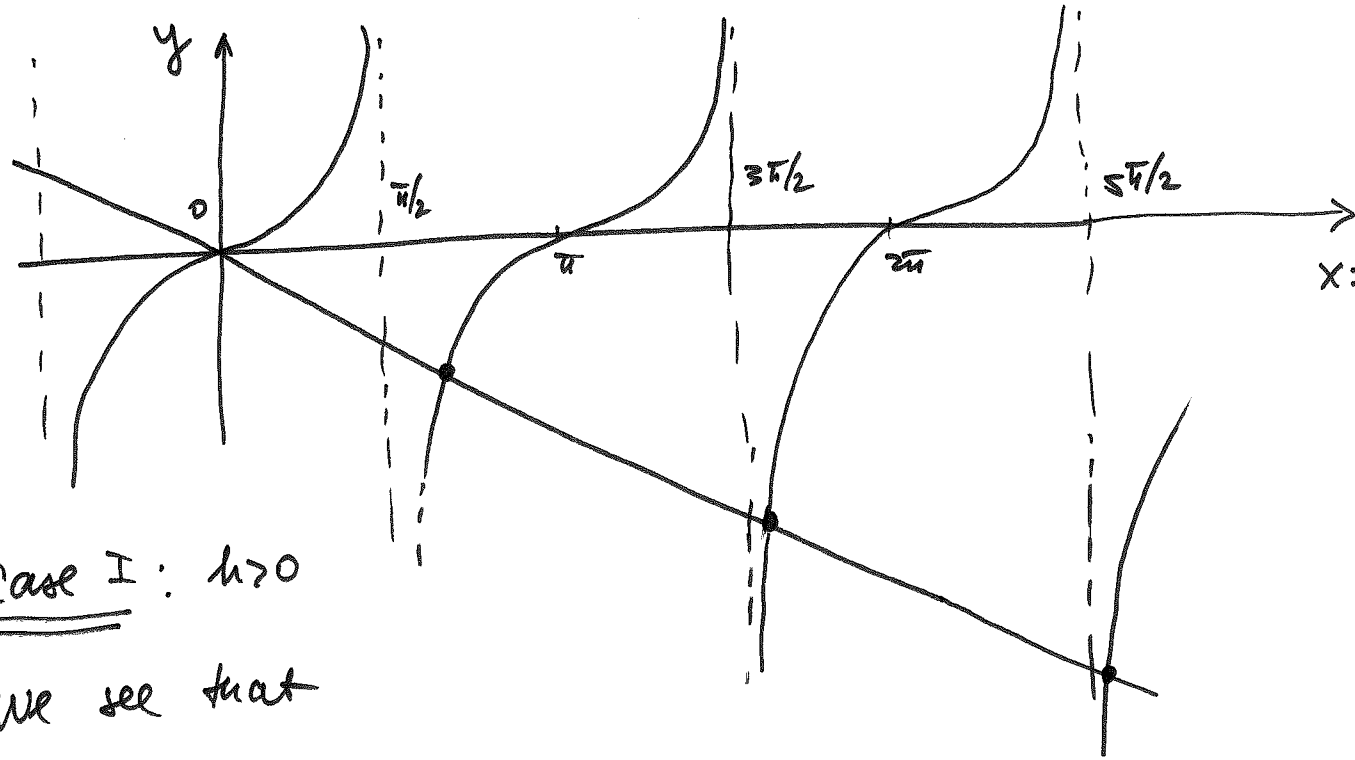
$\lambda > 0$



<u>Case I</u> : $h > 0$

We see that

$\frac{\pi}{2} < \sqrt{\lambda_1} L < \pi$

$h > 0 \Rightarrow \lambda > 0$ by R.-Q.

$\frac{3\pi}{2} < \sqrt{\lambda_2} L < 2\pi$

$\cdots$

$\pi(n - \frac{1}{2}) < \sqrt{\lambda_n} L < n\pi$

as $n \to \infty$ $\sqrt{\lambda} L \sim (n - \frac{1}{2})\pi$

Case II : $h<0$

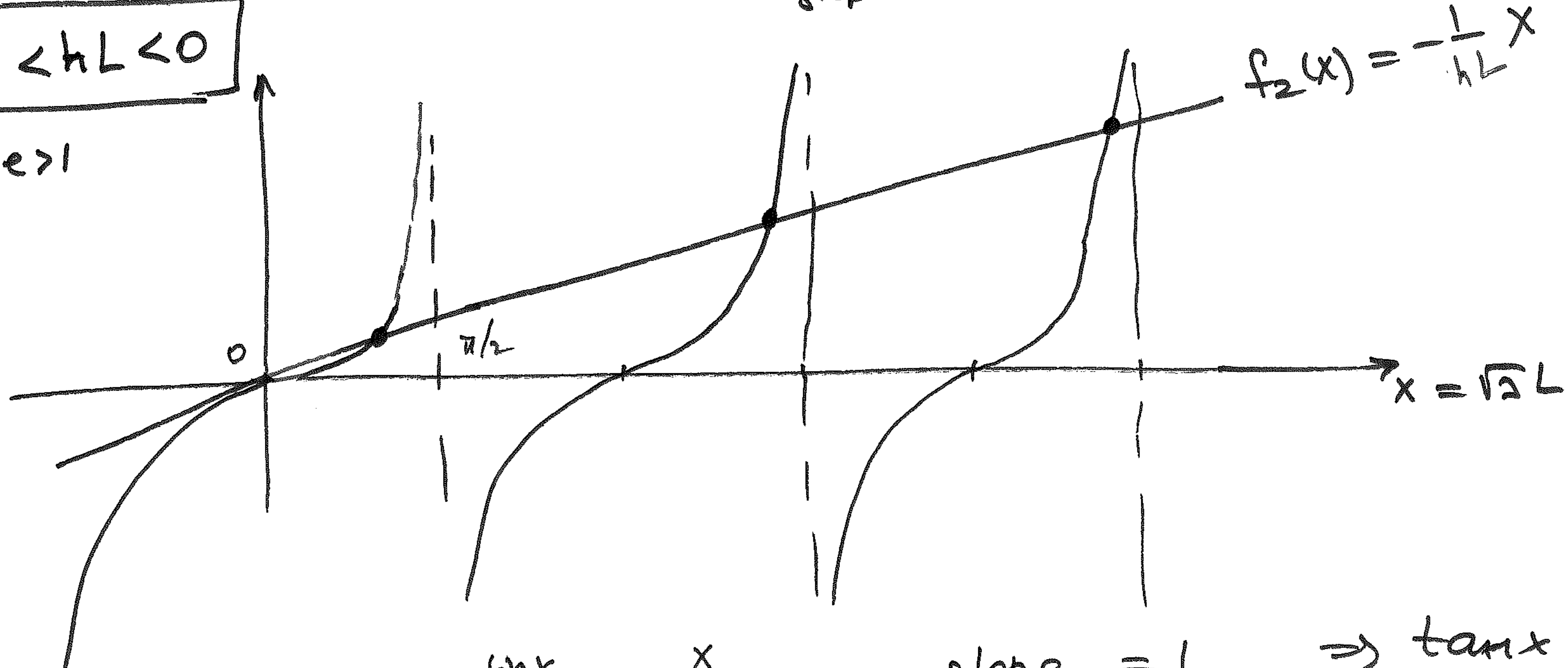
We need to find for which h e'value $\lambda>0$.

We still have $\tan\sqrt{\lambda}L = -\frac{1}{h}\sqrt{\lambda}$

but now we have $f_2(x) = \underbrace{-\frac{1}{hL}}_{\text{slope}} x$ has positive slope $-\frac{1}{hL}$

$\boxed{-1<hL<0}$

Slope>1



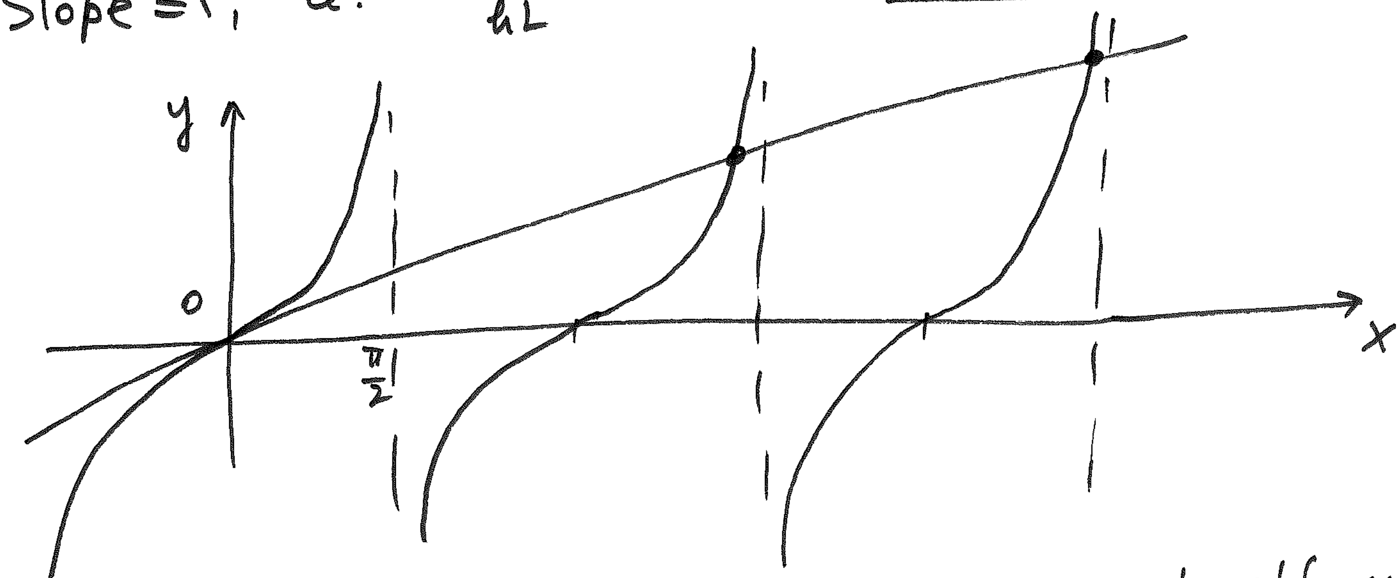
$x \ll 1$ $\quad \tan x = \frac{\sin x}{\cos x} \sim \frac{x}{1} = x$: slope $=1$ $\Rightarrow$ tan x has slope 1 at $x=0$

$f_2(x) = -\frac{1}{hL}x = -\frac{1}{h}\sqrt{\lambda}$ $\qquad$ slope >1 $\Rightarrow$ $-\frac{1}{hL} > 1$

$-\frac{1}{hL} > 1 \implies \boxed{-1 < hL < 0}$ (since $h<0$)

We have ∞ many positive e'values w/ e'functions $\phi_n(x) = \sin\sqrt{\lambda_n}\,x$

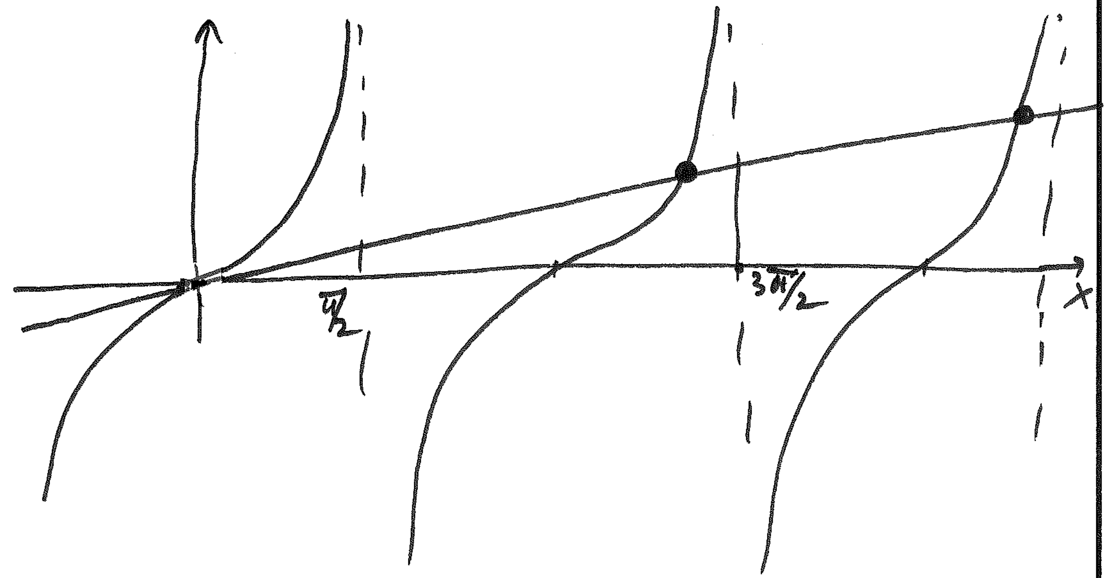
Slope $=1$, i.e. $-\frac{1}{hL} = 1$ or $\boxed{hL = -1}$



Still ∞ many positive e'values λ_n w/ e'functions $\phi_n(x) = \sin\sqrt{\lambda_n}\,x$

Slope < 1

$-\frac{1}{hL} < 1 \Rightarrow \boxed{hL < -1}$



Still ∞ many positive e'values λ_n w/ e'functions

$\phi_n(x) = \sin\sqrt{\lambda_n}\, x$

Zero e'value : $\lambda = 0$

Case $\lambda = 0 \overset{RQ}{\Rightarrow} h < 0$

$\phi'' + \cancel{\lambda}^{0}\phi = 0 \Rightarrow \phi'' = 0 \Rightarrow \phi(x) = ax + \cancel{b}^{0}$

$\phi(0) = 0 \Rightarrow 0 = \phi(0) = a \cdot 0 + b \Rightarrow b = 0 \Rightarrow \phi(x) = ax \;,\; \phi' = a$

$\phi'(L) + h\phi(L) = 0 \Rightarrow a + h \cdot aL = 0$ or $\underset{\neq 0}{a}(1 + hL) = 0$

$\Rightarrow 1 + hL = 0$ or $\boxed{hL = -1}$

Only when $hL = -1$, it is possible to have $\lambda = 0$ w/ e'function $\phi(x) = x$.

Negative e'values: $\lambda<0$ $\quad$ RQ $\Rightarrow$ $h<0$

$\lambda<0$. let $s=-\lambda>0$

$\phi''-s\phi=0$

$\phi(x)=\cancel{C_1}^{\,0}\cosh\sqrt{s}x + C_2\sinh\sqrt{s}x$

$\phi(0)=0 \Rightarrow C_1=0 \Rightarrow \phi(x)=C_2\sinh\sqrt{s}x$

$\phi'(x)=C_2\sqrt{s}\cosh\sqrt{s}x$

$\phi'(L)+h\phi(L)=0$

$\therefore\; C_2\sqrt{s}\cosh\sqrt{s}L + hC_2\sinh\sqrt{s}L=0$ $\qquad C_2\neq 0$

$\sqrt{s}\cosh\sqrt{s}L + h\sinh\sqrt{s}L=0$

or $\boxed{\tanh\sqrt{s}L = -\frac{1}{h}\sqrt{s}}$ $\qquad$ eqn for s

Aside: $\tanh t = \dfrac{\sinh t}{\cosh t} = \dfrac{e^t-e^{-t}}{e^t+e^{-t}}$

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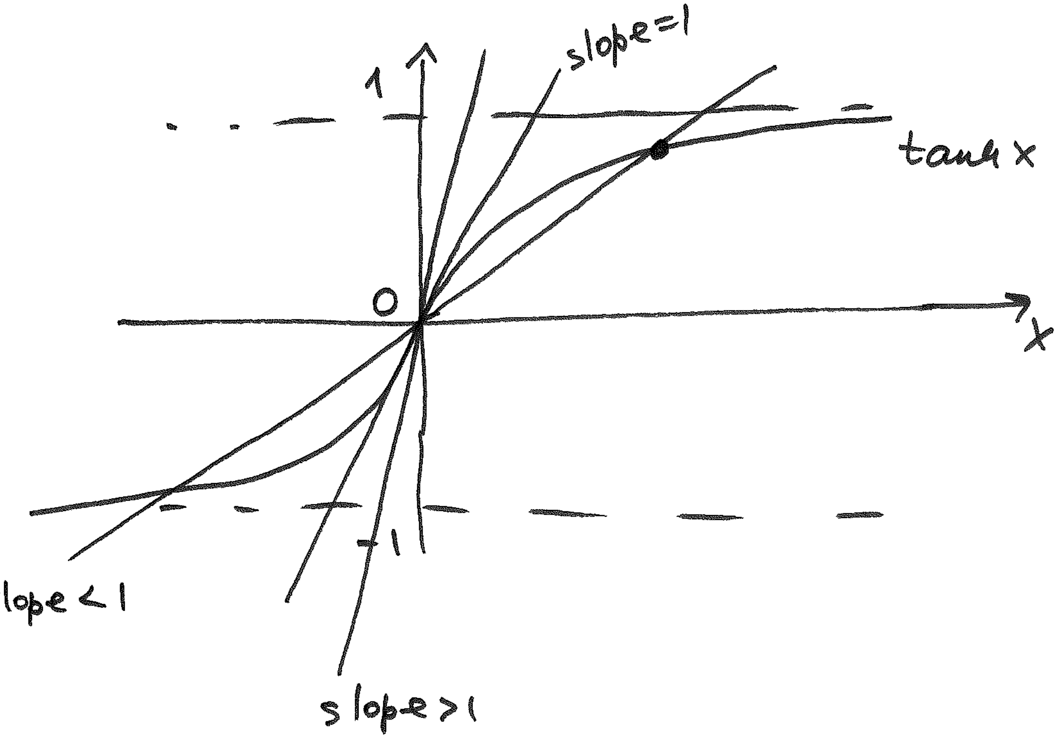
Let $x = \sqrt{s}\, L$

$f_1(x) = \tanh x$

$-\frac{1}{h}\sqrt{s} = \underbrace{-\frac{1}{hL}}_{\text{slope} > 0} x = f_2(x) \qquad h<0$



Slope of $\tanh x$ at $x=0$ is 1

Slope >1 i.e. $-\frac{1}{hL} > 1 \Rightarrow \boxed{-1 < hL < 0}$ since $h<0$

(no roots, no e'values)

Slope $=1$ $\quad -\frac{1}{hL} = 1 \Rightarrow \boxed{hL = -1}$

no roots, no e'values

Slope <1 i.e. $-\frac{1}{hL} < 1 \Rightarrow \boxed{hL < -1}$

We have one root and e'value $\lambda < 0$.

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Summary

1. $h \geq 0 \Rightarrow \lambda_n > 0 \qquad \phi_n(x) = \sin(\sqrt{\lambda_n}\, x), \qquad n > 0$

2. $-1 < hL < 0 \Rightarrow \lambda_n > 0, \qquad \phi_n(x) = \sin(\sqrt{\lambda_n}\, x), \qquad n > 0$

3. $hL = -1 \Rightarrow$
   $\lambda_1 = 0 \qquad \phi_1(x) = x$
   $\lambda_n > 0 \qquad \phi_n(x) = \sin(\sqrt{\lambda_n}\, x), \qquad n > 1$

4. $hL < -1 \Rightarrow$
   $\lambda_1 < 0 \qquad \phi_1(x) = \sinh(\sqrt{-\lambda_1}\, x)$
   $\lambda_n > 0 \qquad \phi_n(x) = \sin(\sqrt{\lambda_n}\, x), \qquad n > 1$

a closer look at $hL < -1$ case (unphysical case):

$u_t = k u_{xx} \qquad 0 < x < 1$

$u(0,t) = 0 \qquad u_x(L,t) + h u(L,t) = 0 \qquad hL < -1$

$$G(t) = Ce^{-k\lambda t} \qquad \phi_n(x) = \begin{cases} \sinh\left(\sqrt{-\lambda_1}\, x\right) \\ \sin\left(\sqrt{\lambda_n}\, x\right) \end{cases}$$

$$u(x,t) = a_1 e^{-k\lambda_1 t} \sinh\left(\sqrt{-\lambda_1}\, x\right) + \sum_{n=2}^{\infty} a_n e^{-k\lambda_n t} \sin\left(\sqrt{\lambda_n}\, x\right)$$

IC: $$\underset{\substack{\parallel \\ f(x)}}{u(x,0)} = a_1 \sinh\left(\sqrt{-\lambda_1}\, x\right) + \sum_{n=2}^{\infty} a_n \sin\left(\sqrt{\lambda_n}\, x\right)$$

where

$$a_1 = \frac{\int_0^1 \sinh\left(\sqrt{-\lambda_1}\, x\right) f(x)\, dx}{\int_0^1 \sinh^2\left(\sqrt{-\lambda_1}\, x\right) dx} \qquad a_n = \frac{\int_0^1 \sin\left(\sqrt{\lambda_n}\, x\right) f(x)\, dx}{\int_0^1 \sin^2\left(\sqrt{\lambda_n}\, x\right)}, \quad n > 1$$

$u(x,t) \sim a_1 e^{-k\lambda_1 t} \sinh\left(\sqrt{-\lambda_1}\, x\right)$ for large t

since $\lambda_1 < 0$, $\lim_{t \to \infty} |u(x,t)| = \infty$