Lecture 35

ϕ-problem (spatial)

$p=1,\ q=0,\ \sigma=1$: regular S.-L. problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

$$\phi(0) = 0, \qquad \phi'(L) = -h\,\phi(L) \qquad \text{or} \qquad \phi'(L) + h\,\phi(L) = 0, \quad h \neq 0$$

Rayleigh Quotient:

$$\lambda = \frac{-\underset{=1}{p}\,\phi\phi'\Big|_0^L + \int_0^L \left[\underset{=1}{p}(\phi')^2 - \cancelto{0}{q}\,\phi^2\right] dx}{\int_0^L \phi^2 \underset{=1}{\sigma}\, dx}$$

$$-\underset{=1}{p}\,\phi\phi'\Big|_0^L = -\phi\phi'\Big|_0^L = -\phi(L)\underbrace{\phi'(L)}_{-h\phi(L)} + \cancelto{0}{\phi(0)}\,\phi'(0) = h\,\phi^2(L)$$

$$\therefore \quad \lambda = \frac{\overbrace{h\,\phi^2(L)}^{>0} + \overbrace{\int_0^L (\phi')^2\,dx}^{\geq 0}}{\underbrace{\int_0^L \phi^2\,dx}_{>0}}, \qquad h \neq 0$$

1. if $h>0 \Rightarrow \lambda>0$
2. if $h<0 \Rightarrow \lambda<0$ if $|h\,\phi^2(L)| > \int_0^L (\phi')^2\,dx$

<u>Positive e'values</u> : $\lambda>0$

If $h>0 \Rightarrow \lambda>0$ (from RQ)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad \phi(0)=0, \qquad \phi'(L) = -h\,\phi(L)$$

$\phi(x) = C_1 \cos\sqrt{\lambda}\, x + C_2 \sin\sqrt{\lambda}\, x$

$\phi(0)=0 \Rightarrow C_1 = 0 \Rightarrow \phi(x) = C_2 \sin\sqrt{\lambda}\, x$

$\phi'(x) = C_2 \sqrt{\lambda} \cos\sqrt{\lambda}\, x$

$\phi'(L) + h\,\phi(L) = 0 \Rightarrow C_2\sqrt{\lambda}\cos\sqrt{\lambda}L + h\, C_2 \sin\sqrt{\lambda}L = 0$

$C_2 (\sqrt{\lambda}\cos\sqrt{\lambda}L + h \sin\sqrt{\lambda}L) = 0$

$\neq 0$

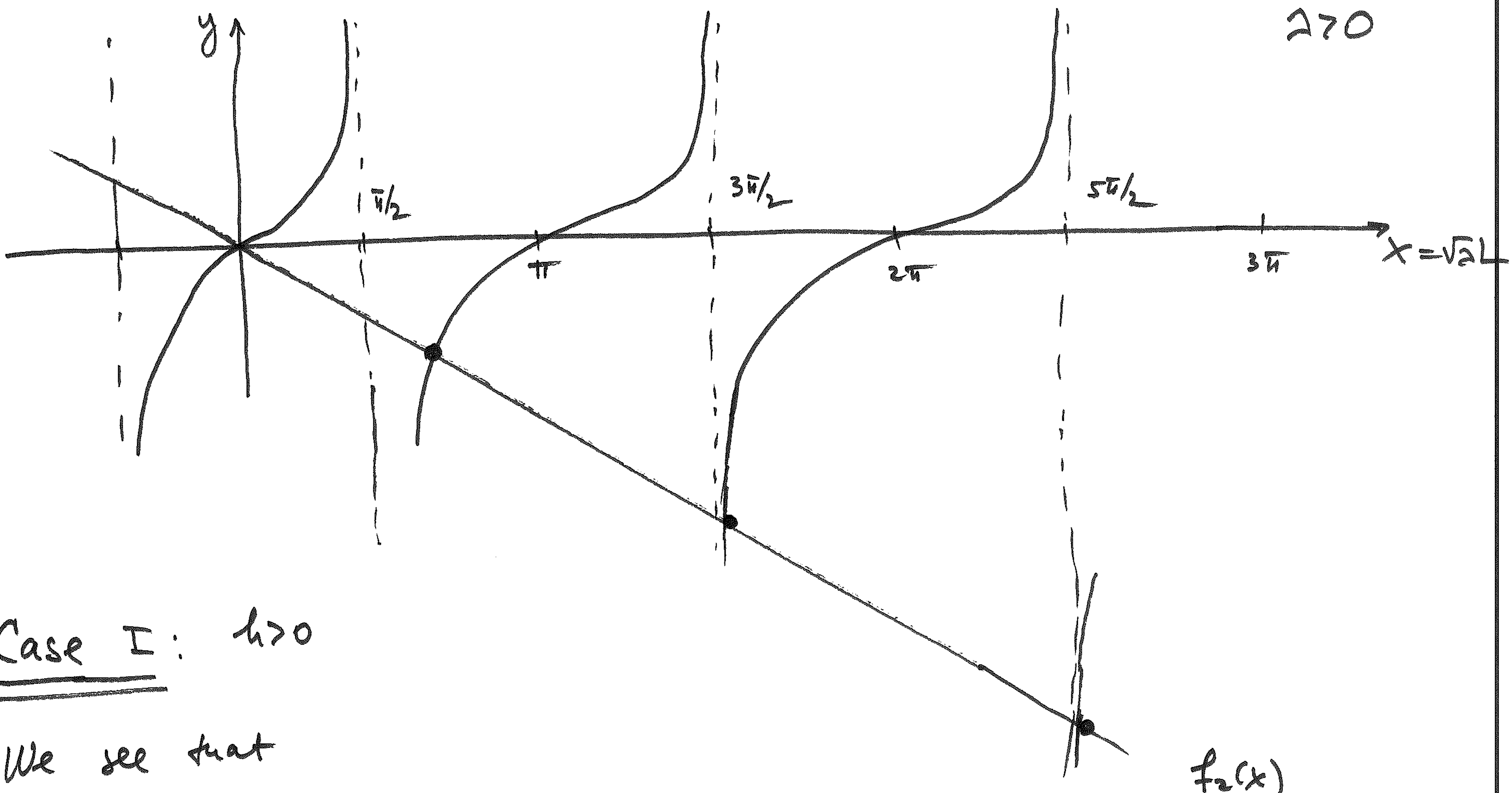
$\therefore \quad \sqrt{\lambda}\cos\sqrt{\lambda}L + h\sin\sqrt{\lambda}L = 0 \qquad \Big| \dfrac{1}{h\cos\sqrt{\lambda}L}$

$\boxed{\tan\sqrt{\lambda}L = -\frac{1}{h}\sqrt{\lambda}}$ transcendental eq^{n}

Let $x = \sqrt{\lambda}L$. Denote $f_1(x) = \tan x$

$-\frac{1}{h}\sqrt{\lambda} = -\frac{1}{hL}\underbrace{\sqrt{\lambda}L}_{=x} = -\frac{1}{hL}x \equiv f_2(x)$: linear function w/ slope $-\frac{1}{hL}$

$\lambda > 0$



<u>Case I</u>: $h > 0$

We see that

$\frac{\pi}{2} < \sqrt{\lambda_1} L < \pi$ $\qquad \frac{1}{2}$

$\frac{3\pi}{2} < \sqrt{\lambda_2} L < 2\pi$ $\qquad \frac{3}{2}$

$- \; - \; - \; \cdot \; -$ $\qquad \frac{5}{2}$

$2n-1$ $\qquad n=1 \Rightarrow 1$

$n=2 \Rightarrow 3$

$\frac{2n-1}{2}\pi$

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$\pi(n-\frac{1}{2}) < \sqrt{\lambda_n}\, L < n\pi$

as $n \to \infty$ $\quad \sqrt{\lambda_n}\, L \to (n-\frac{1}{2})\pi$

Case II : $h < 0$

We need to find for which h e'value $\lambda > 0$.

We still have $\quad \tan\sqrt{\lambda}\, L = -\frac{1}{h}\sqrt{\lambda}$

but now we have $\quad f_2(x) = -\frac{1}{hL}\, x \quad$ has positive slope $-\frac{1}{hL}$

$\boxed{-1 < hL < 0}$

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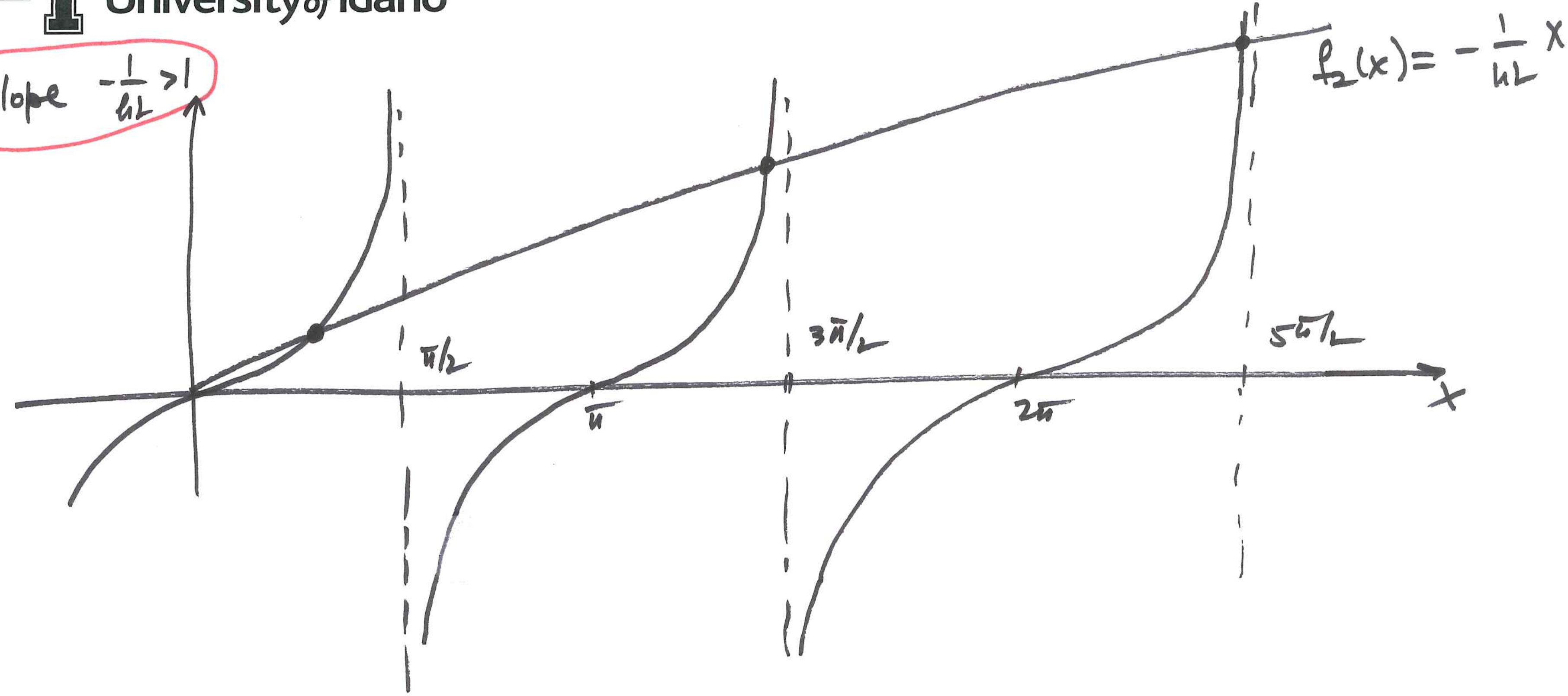


$x \ll 1$
(x is very small)

$\tan x = \frac{\sin x}{\cos x} \sim \frac{x}{1} = x$ $\Rightarrow$ slope of $\tan x$ at $x = 0$ is 1

$f_2(x) = -\frac{1}{hL}x$

slope $= -\frac{1}{hL} > 1$

$-\frac{1}{hL} > 1 \quad | \cdot hL$

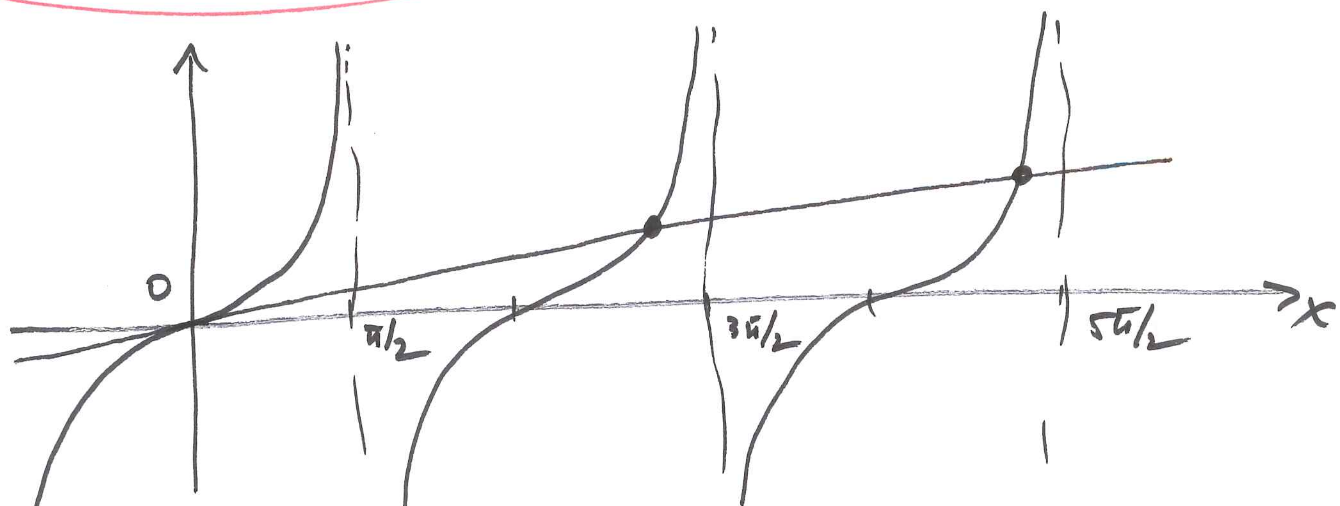
$\boxed{-1 < hL < 0}$

since $h < 0$

We have ∞ many roots / positive e'values w/

e' functions $\phi_n(x) = \sin\sqrt{\lambda_n}\, x$

Slope $-\frac{1}{hL} = 1 \Rightarrow \boxed{hL = -1}$



Still ∞ many positive e'values λ_n w/ e'functions

$\phi_n(x) = \sin\sqrt{\lambda_n}\, x$

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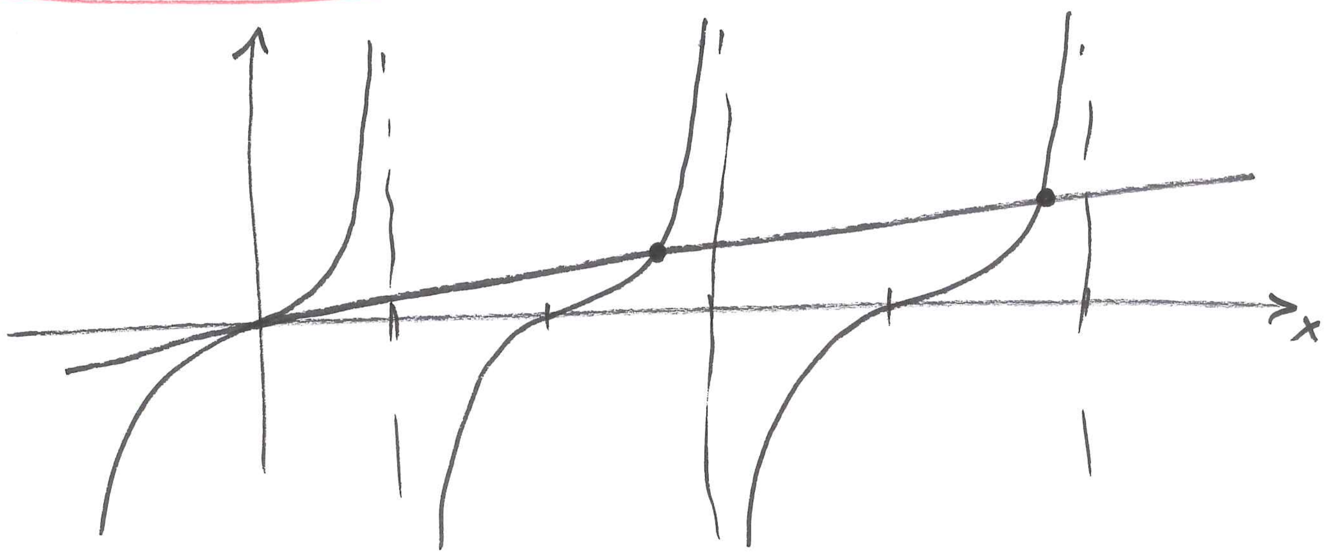
Slope $-\frac{1}{hL} < 1 \quad | \, hL \quad \Rightarrow \quad -1 > hL$ or $\boxed{hL < -1}$



Still ∞ many positive e'values λ_n w/ e'functions

$\phi_n(x) = \sin\sqrt{\lambda_n}\, x$.

Zero e'value : $\lambda = 0$

Case $\lambda = 0$

$$\lambda = \frac{h \overbrace{\Phi^2(L)}^{>0} + \overbrace{\int_0^L (\phi')^2 dx}^{\geq 0}}{\int_0^L \phi^2 dx} = 0 \Rightarrow h < 0$$

$$\phi'' + \cancelto{0}{\lambda} \phi = 0 \Rightarrow \phi'' = 0 \Rightarrow \phi(x) = ax + \cancelto{0}{b}$$

$$\phi(0) = 0 \Rightarrow a \cdot 0 + b = 0 \Rightarrow b = 0 \Rightarrow \phi(x) = ax$$

$$\phi'(x) = a$$

$$\phi'(L) + h\,\Phi(L) = 0 \Rightarrow a + h \cdot aL = 0 \Rightarrow \underset{\neq 0}{a}(1 + hL) = 0$$

$$\Rightarrow 1 + hL = 0 \Rightarrow hL = -1$$

Only when $hL = -1$, it is possible to have e'value $\lambda = 0$ w/ e'function $\phi(x) = x$.

Negative e'values : $\lambda < 0$

RQ $\Rightarrow$ $h < 0$

$$\phi'' + \lambda \phi = 0$$

let $s = -\lambda > 0$

$$\phi'' - s\phi = 0$$

$$\phi(x) = C_1 \cosh \sqrt{s}\, x + C_2 \sinh \sqrt{s}\, x$$

$$\phi(0) = 0 \Rightarrow C_1 \underbrace{\cosh 0}_{=1} + C_2 \underbrace{\sinh 0}_{=0} = 0 \Rightarrow C_1 = 0$$

$\phi(x) = C_2 \sinh \sqrt{s}\, x$

$\phi'(x) = C_2 \sqrt{s} \cosh \sqrt{s}\, x$

$\phi'(L) + h\,\phi(L) = 0 \Rightarrow C_2 \sqrt{s} \cosh \sqrt{s}\, L + h\, C_2 \sinh \sqrt{s}\, L = 0$

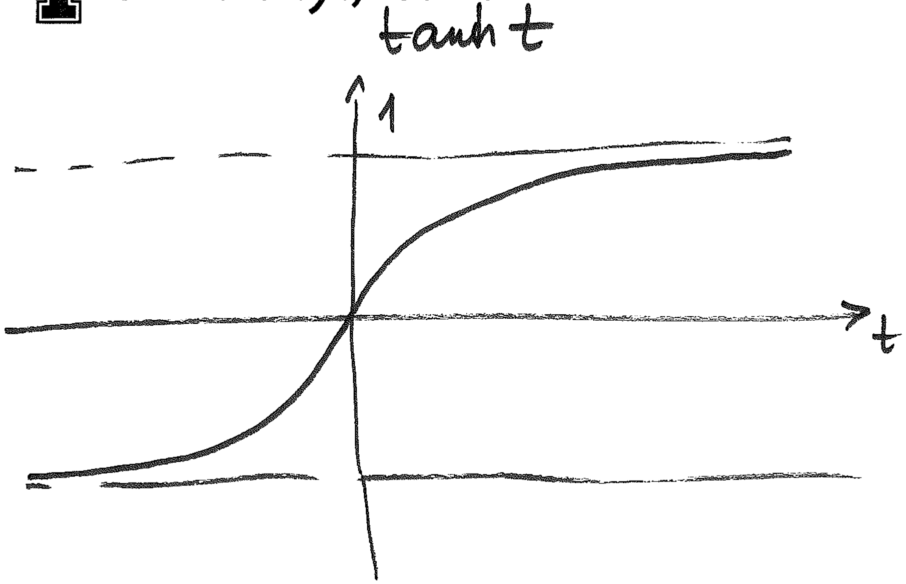
$\underset{\neq 0}{C_2} \left(\sqrt{s} \cosh \sqrt{s}\, L + h \sinh \sqrt{s}\, L \right) = 0$

$\sqrt{s} \cosh \sqrt{s}\, L + h \sinh \sqrt{s}\, L = 0$

or $\boxed{\tanh \sqrt{s}\, L = -\frac{1}{h} \sqrt{s}}$ eqn for s

<u>Recall</u>

$$\tanh t = \frac{\sinh t}{\cosh t} = \frac{\frac{e^t - e^{-t}}{2}}{\frac{e^t + e^{-t}}{2}} = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$



Now $x = \sqrt{s}\,L$

$$\tanh x = -\frac{1}{hL}\,x$$