Lecture 36

## Large eigenvalues of Sturm-Liouville problem (asymptotic formula)

Recall Sturm-Liouville equation:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + [q(x) + \lambda\sigma(x)]\,\phi(x) = 0 \qquad a \le x \le b$$

e'values: $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$

$$\lim_{n\to\infty} \lambda_n = \infty$$

$\lambda_1$ is the smallest e'value. We can use RQ to approximate small e'values.

It turns out that we can approximate large e'values asymptotically.

Let $\lambda \gg 1$ ($\lambda$ is very large)

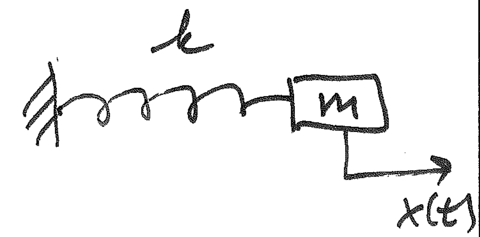
$$\Rightarrow \lambda \sigma(x) \gg q(x)$$

Then S.-L. equation reduces to

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + \lambda\sigma(x)\,\phi(x) = 0$$

$$\underbrace{p(x)}_{m}\frac{d^2\phi}{dx^2} + \underbrace{\frac{dp}{dx}}_{c}\frac{d\phi}{dx} + \underbrace{\lambda\sigma(x)}_{k}\,\phi = 0 \qquad (1)$$

Think about S.-L. equation as of an equation for mass-spring system w/ damping where time is being replaced w/ $x$ and displacement $x(t)$ — with $\phi(x)$




Mass-spring system:

$x(t)$: displacement from equilibrium

$m$: mass

$k$: spring constant

$c$: damping coefficient

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$$m\ddot{x} + c\dot{x} + kx = 0$$

The restoring force $F(x) = -kx$. In our case, this force is $-\lambda\sigma(x)\phi(x)$. Since $\lambda$ is large, the restoring force is also large $\Rightarrow$ $\phi(x)$ is rapidly oscillating function. This is in agreement with the fact that $\phi(x)$ has many zeros for large $\lambda$.

Since $\phi(x)$ is rapidly oscillating function, functions $p(x)$ and $\sigma(x)$ are slowly varying functions compared to $\phi(x)$.

$$\therefore \quad p(x) \approx p(x_0) \qquad \sigma(x) \approx \sigma(x_0)$$

$$\Downarrow$$

$$\frac{dp}{dx} \approx 0$$

for all $x$ near any pt $x_0$. Hence, eq$^{\underline{n}}$ (1) can be written (approximated) as

$$\underbrace{p(x_0)}_{\text{const}} \frac{d^2\phi}{dx^2} + \underbrace{\lambda \sigma(x_0)}_{\text{const}} \phi(x) = 0 \qquad \text{near any pt } x_0$$

$m \qquad\qquad k$

Recall solution to a mass-spring system eq$^n$:

$$m\ddot{x} + kx = 0 \qquad\qquad x = e^{rx}$$

$$\ddot{x} + \frac{k}{m}x = 0 \qquad\qquad \text{let } \omega_0^2 = \frac{k}{m}$$

Characteristic equation: $r^2 + \frac{k}{m} = 0$

Then $\omega_0 = \sqrt{\frac{k}{m}}$ and $r = \pm i\omega_0$

$\omega_0$: angular frequency / local frequency

$$x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$$

or $x(t) = A \sin(\omega_0 t - \alpha)$ (2)

where $A = \sqrt{C_1^2 + C_2^2}$ : amplitude

$\alpha$: phase shift

In our case, $m = p(x_0)$, $k = 2\sigma(x_0)$. Then the angular / local frequency can be written as

$$\text{"}\omega_0\text{"} = \begin{array}{c}\text{local}\\ \text{frequency}\end{array} = \sqrt{\frac{2\sigma(x_0)}{p(x_0)}}$$

If we use this expression for frequency, then $\phi(x)$ will be an oscillatory function w/ constant amplitude like in (2).

A better approximation to $\phi(x)$ will be if we assume that

$$\phi(x) = A(x)\sin(\psi(x))$$

where
$A(x)$: slowly varying amplitude
$\psi(x)$: phase

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Expand $\psi(x)$ in a Taylor series about $x = x_0$:

$$\psi(x) = \psi(x_0) + \psi'(x_0)(x - x_0) + \dots$$

$$\Rightarrow \sin(\psi(x)) = \sin(\underline{\psi'(x_0)}(x - x_0) + \psi(x_0) + \dots)$$

compare this with $\sin(\underline{\omega_0} t - \alpha)$

$\therefore$ $\phi(x)$ oscillates w/ local frequency given by

$$\psi'(x_0) = \sqrt{\frac{\lambda \sigma(x_0)}{p(x_0)}} \qquad a \le x \le b$$

Then $$\psi(x) = \int_a^x \sqrt{\frac{\lambda \sigma(s)}{p(s)}}\, ds + \psi(a) \quad : \text{exact}$$

Check: $$\psi'(x) = \sqrt{\frac{\lambda \sigma(x)}{p(x)}}$$

$$\psi(a) = \psi(a) \qquad \checkmark$$

For large $\lambda$,

$$\Psi(x) \approx \int_a^x \left(\frac{\lambda\,\sigma(s)}{p(s)}\right)^{1/2} ds$$

It can be shown that

$$A(x) = [\sigma(x)p(x)]^{-1/4} \quad \text{(to the leading order)}$$

$$\therefore \boxed{\phi(x) = [\sigma(x)p(x)]^{-1/4} \sin \int_a^x \left[\frac{\lambda\,\sigma(s)}{p(s)}\right]^{1/2} ds}$$

approximation of an e'function $\phi(x)$ for large $\lambda$

To find $\lambda$, we need to use BCs.

For example,

$$\phi(0) = \phi(L) = 0$$

$$0 = \phi(L) = \left[\sigma(L)p(L)\right]^{-1/4} \sin \int_0^L \left[\frac{\lambda \sigma(s)}{p(s)}\right]^{1/2} ds$$

$$a = 0$$

$$\therefore \quad \sin \int_0^L \left[\frac{\lambda \sigma(s)}{p(s)}\right]^{1/2} ds = 0$$

$$\int_0^L \left[\frac{\lambda \sigma(s)}{p(s)}\right]^{1/2} ds = n\pi, \qquad n = 1, 2, \dots$$

$$\lambda^{1/2} \int_0^L \left[\frac{\sigma(s)}{p(s)}\right]^{1/2} ds = n\pi, \qquad n = 1, 2, \dots$$

$$\therefore \quad \boxed{\lambda = \frac{(n\pi)^2}{\left\{\int_0^L \left[\frac{\sigma(s)}{p(s)}\right]^{1/2} ds\right\}^2}}$$

approximation of large e'values $\lambda_n$

Ex $\phi'' + \lambda(1+x)\phi = 0$ $\qquad p = 1, \quad q(x) = 0, \quad \sigma(x) = 1+x$

$\phi(0) = \phi(1) = 0$

Hence,

$$\lambda \approx \frac{n^2\pi^2}{\left[\int_0^1 \left(\frac{1+s}{1}\right)^{1/2} ds\right]^2} = \frac{n^2\pi^2}{\frac{2}{3}(1+s)^{3/2}\Big|_0^1} = \boxed{\frac{n^2\pi^2}{\frac{4}{9}\left(2^{3/2}-1\right)^2}}$$

gives very good approximation for small e'values

TABLE 5.9.1: Eigenvalues $\lambda_n$

| $n$ | Numerical answer* (assumed accurate) | Asymptotic formula (5.9.11) | Error | % error |
|---|---|---|---|---|
| 1 | 6.548395 | 6.642429 | 0.094034 | 1.436 |
| 2 | 26.464937 | 26.569718 | 0.104781 | 0.3950 |
| 3 | 59.674174 | 59.781865 | 0.107691 | 0.1805 |
| 4 | 106.170023 | 106.278872 | 0.108849 | 0.1025 |
| 5 | 165.951321 | 166.060737 | 0.109416 | 0.0659 |
| 6 | 239.0177275 | 239.1274615 | 0.109734 | 0.0459 |
| 7 | 325.369115 | 325.479045 | 0.109930 | 0.0338 |

*Courtesy of E. C. Gartland, Jr.

## ERCISES 5.9

**5.9.1.** Estimate (to leading order) the large eigenvalues and corresponding eigenfunctions for

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + [\lambda\sigma(x) + q(x)]\phi = 0$$

if the boundary conditions are

(a) $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

*(b) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

(c) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) + h\phi(L) = 0$

**5.9.2.** Consider

$$\frac{d^2\phi}{dx^2} + \lambda(1+x)\phi = 0$$

subject to $\phi(0) = 0$ and $\phi(1) = 0$. Roughly sketch the eigenfunctions for $\lambda$ large. Take into account amplitude and period variations.

**5.9.3.** Consider for $\lambda \gg 1$

$$\frac{d^2\phi}{dx^2} + [\lambda\sigma(x) + q(x)]\phi = 0.$$

*(a) Substitute

## Eigenvalues $\lambda_n$: comparison between numerically and asymptotically computed eigenvalues

| $\lambda_n^{numer}$ | $\lambda_n^{asympt}$ | absolute error= $= \lvert\lambda_n^{numer} - \lambda_n^{asympt}\rvert$ | relative error= $= \frac{\lvert\lambda_n^{numer} - \lambda_n^{asympt}\rvert}{\lambda_n^{numer}}$ |
|---|---|---|---|
| 6.548395 | 6.642429 | 0.094034 | 0.014360 |
| 26.464937 | 26.569718 | 0.104781 | 0.003959 |
| 59.674174 | 59.781865 | 0.107691 | 0.001805 |
| 106.170023 | 106.278872 | 0.108849 | 0.001025 |
| 165.951321 | 166.060737 | 0.109416 | 0.000659 |
| 239.017728 | 239.127462 | 0.109734 | 0.000459 |
| 325.369115 | 325.479045 | 0.109930 | 0.000338 |

Note: The relative error (percent error) decreases as $n$ increases where the absolute error stays about the same. This approximately constant error can be improved by adding a correction term to the asymptotic formula for eigenvalues.