University of Idaho

Lecture 36

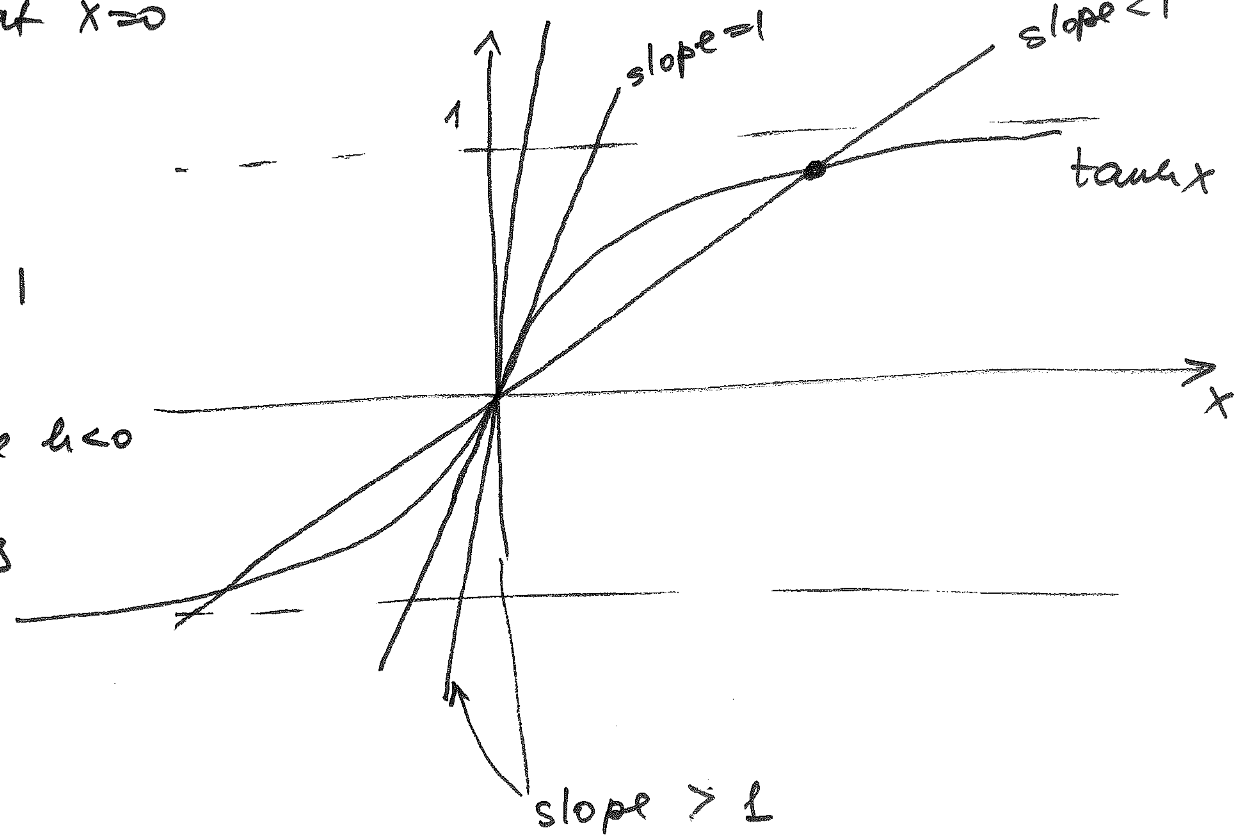
Let $x = \sqrt{s}\, L$. Then

$f_1(x) = \tanh x$

$$f_2(x) = -\frac{1}{h}\sqrt{s} = -\frac{1}{hL}\underbrace{\sqrt{s}L}_{x} = -\underbrace{\frac{1}{hL}}_{>0}\, x$$

positive slope

Slope of $\tanh x$ at $x=0$ is 1



* Slope > 1 $\Rightarrow -\frac{1}{hL} > 1$

$\boxed{-1 < hL < 0}$ since $h<0$

no roots, no e'values

* Slope $=1 \Rightarrow -\frac{1}{hL} = 1 \Rightarrow \boxed{hL = -1}$

no roots, no e'values

* Slope $<1 \Rightarrow -\frac{1}{hL} < 1 \Rightarrow -1 > hL$ or $\boxed{hL < -1}$

We have one root and one e'value $\lambda < 0$.

Summary

1. $h \geq 0 \Rightarrow \lambda_n > 0$ $\qquad \phi_n(x) = \sin(\sqrt{\lambda_n}\, x), \quad n > 0$

2. $-1 < hL < 0 \Rightarrow \lambda_n > 0$ $\qquad \phi_n(x) = \sin(\sqrt{\lambda_n}\, x), \quad n > 0$

3. $hL = -1 \Rightarrow$ $\quad \lambda_1 = 0 \qquad \phi_1(x) = x$

   $\lambda_n > 0 \qquad \phi_n(x) = \sin(\sqrt{\lambda_n}\, x), \quad n > 1$

4. $hL < -1 \Rightarrow \lambda_1 < 0 \qquad \phi_1(x) = \sinh\left(\sqrt{-\lambda_1}\, x\right)$

$\lambda_n > 0 \qquad \phi_n(x) = \sin\left(\sqrt{\lambda_n}\, x\right), \quad n > 1$

a closer look at $hL < -1$ case (unphysical case)

$u_t = k u_{xx} \qquad 0 < x < 1$

$u(0,t) = 0 \qquad u_x(L,t) + h\, u(L,t) = 0$

$G(t) = C e^{-k\lambda t} \qquad \phi_n(x) = \begin{cases} \sinh\left(\sqrt{-\lambda_1}\, x\right) \\ \sin\left(\sqrt{\lambda_n}\, x\right) \end{cases}$

$$u(x,t) = a_1 e^{-k\lambda_1 t} \sinh\left(\sqrt{-\lambda_1}\, x\right) + \sum_{n=2}^{\infty} a_n e^{-k\lambda_n t} \sin\left(\sqrt{\lambda_n}\, x\right)$$

IC: $$u(x,0) = a_1 \sinh\left(\sqrt{-\lambda_1}\, x\right) + \sum_{n=2}^{\infty} a_n \sin\left(\sqrt{\lambda_n}\, x\right)$$

$$\parallel$$

$$f(x)$$

where

$$a_1 = \frac{\int_0^1 \sinh\left(\sqrt{-\lambda_1}\, x\right) f(x)\, dx}{\int_0^1 \sinh^2\left(\sqrt{-\lambda_1}\, x\right) dx} \qquad a_n = \frac{\int_0^1 \sin\left(\sqrt{\lambda_n}\, x\right) f(x)\, dx}{\int_0^1 \sin^2\left(\sqrt{\lambda_n}\, x\right) dx}$$

$u(x,t) \sim a_1 e^{-k\lambda_1 t} \sinh\left(\sqrt{-\lambda_1}\, x\right)$ for large t

since $\lambda_1 < 0 \Rightarrow -\lambda_1 > 0 \Rightarrow |u(x,t)| \xrightarrow[t\to\infty]{} \infty$

Large eigenvalues of Sturm-Liouville problem (asymptotic formula)

Recall Sturm-Liouville problem

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + \left[q(x) + \lambda\sigma(x)\right]\phi(x) = 0 \qquad a \le x \le b$$

E'values: $\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots$

$$\lim_{n\to\infty} \lambda_n = \infty$$

λ_1 is the smallest e'value. We can use RQ to approximate small e'values.

We can approximate large e'values asymptotically.

Let $\lambda \gg 1$ (λ is very large)

$\therefore \lambda \sigma(x) \gg q(x)$

Then S.-L. eqn reduces to

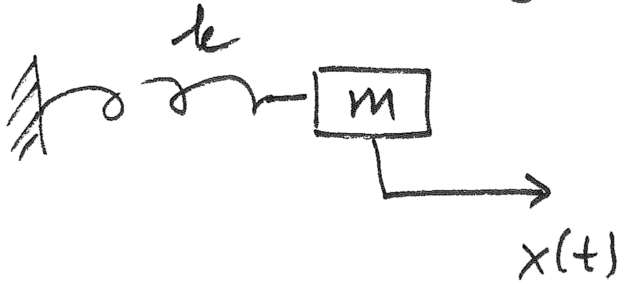
$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + \lambda\sigma(x)\,\phi(x) = 0$$

$$\underbrace{p(x)}_{m}\frac{d^2\phi}{dx^2} + \underbrace{\frac{dp}{dx}}_{c}\cdot\frac{d\phi}{dx} + \underbrace{\lambda\sigma(x)}_{k}\,\phi = 0 \qquad (1) \qquad \dot{} = \frac{d}{dt}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

Think about S.-L. equation as an eqn for mass-spring system w/ damping where time t is replaced w/ x and displacement $x(t)$ is replaced w/ e'function $\phi(x)$.

Recall mass-spring systems w/ damping.



$x(t)$: displacement from equilibrium position

k: spring constant

c: damping coefficient

$m\ddot{x} + c\dot{x} + kx = 0$

The restoring force is $F(x) = -kx$. In the case of S.-L. eq^n, the restoring force is $-\lambda\sigma(x)\phi(x)$. Since e'value λ is large, the restoring force is also large

$\Rightarrow \phi(x)$ is a rapidly oscillating function. This is in agreement that $\phi(x)$ has many zeros for large λ

(large $\lambda \Rightarrow \lambda = \lambda_n$, i.e. large index $n \Rightarrow \phi_n(x)$ has $n-1$ roots in (a,b).)

Since $\phi(x)$ is a rapidly oscillating function, functions $p(x)$ and $\sigma(x)$ are slowly varying functions compared to $\phi(x)$.

$$\therefore \quad p(x) \approx p(x_0) \qquad \sigma(x) \approx \sigma(x_0)$$

for all x near any pt x_0. Then

$$\frac{dp}{dx} \approx 0$$

Hence, eq^{n} (1) can be approximated by

$$\underbrace{p(x_0)}_{\substack{\text{const} \\ m}} \frac{d^2\phi}{dx^2} + \underbrace{\lambda\sigma(x_0)}_{\substack{\text{const} \\ k}} \phi = 0 \qquad (2) \quad \text{near any pt } x_0$$

Recall solution to the mass-spring system w/o damping

$$m\ddot{x} + kx = 0 \qquad k>0,\ m>0$$

$$m\left(\ddot{x} + \frac{k}{m}x\right) = 0 \qquad \omega_0^2 = \frac{k}{m}$$

$$\ddot{x} + \omega_0^2 x = 0$$

ω_0: natural frequency (angular frequency)

Assume $x(t) = e^{rt} \Rightarrow r^2 + \omega_0^2 = 0$, $r = \pm i\omega_0$

$$x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t = A\sin(\omega_0 t - \alpha)$$

$A = \sqrt{C_1^2 + C_2^2}$: amplitude

α: phase shift

In our case,

$$m = p(x_0) \qquad k = 2\sigma(x_0)$$

$$"\omega_0" = \sqrt{\frac{k}{m}} = \sqrt{\frac{2\sigma(x_0)}{p(x_0)}}$$

With this frequency and having in mind that $\phi(x)$ satisfies (approximately) eq^{2} (α), we can say that $\phi(x)$ is an oscillatory function w/ constant amplitude and frequency $\sqrt{\frac{2\sigma(x_0)}{p(x_0)}}$, i.e.

$$\phi(x) \approx A \cdot \sin\left(\sqrt{\frac{2\sigma(x_0)}{p(x_0)}}\, x - \alpha\right)$$