Lecture 37

Approximation Properties

We know that any piecewise smooth function can be represented by a generalized Fourier series:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$

By orthogonality (with weight $\sigma(x)$):

$$(1) \qquad a_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx} \; : \; \underline{\text{Fourier coefficients}}$$

What if we write

$$f(x) \sim \sum_{n=1}^{M} \alpha_n \phi_n(x)$$

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i.e. we use only M first e'functions.

<u>Q</u> Which choice of α_n's would give the "best" result?; i.e. the best approximation of $f(x)$?

<u>Q</u> Would α_n's depend on M?

<u>Q</u> What is the error?

First, we need to define what we mean with the "best" approximation?

<u>Best approximation = least error</u>

We could use $\max\limits_{a \le x \le b} \Big| \underbrace{f(x)}_{\text{exact}} - \underbrace{\sum_{n=1}^{M} \alpha_n \phi_n(x)}_{\text{approximation}} \Big| = \text{error}$

Not convenient to use.

Mean-square deviation:

$$\boxed{E \stackrel{\text{def}}{=} \int_a^b \Big[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \Big]^2 \sigma(x)\,dx}$$

Note: $E \geq 0$ and $E = 0$ iff $f(x) = \sum_{n=1}^{M} \alpha_n \phi_n(x)$
(recall $\sigma(x) > 0$)

Think of E as of a function of $\alpha_1, \alpha_2, \ldots, \alpha_M$:

$$E = E(\alpha_1, \alpha_2, \ldots, \alpha_M)$$

Q How do we find critical points of E?

$$\frac{\partial E}{\partial \alpha_i} = 0, \qquad i = 1, \ldots, M$$

$$0 = \frac{\partial E}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i} \int_a^b \Big[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \Big]^2 \sigma(x)\,dx =$$

$$= \int_a^b 2\left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x)\right]\left(-\phi_i(x)\right)\sigma(x)\,dx =$$

Orthogonality: $\int_a^b \phi_n(x)\,\phi_i(x)\,\sigma(x)\,dx = 0$ if $n \neq i$

$$= -2\int_a^b \left[f(x)\,\phi_i(x)\,\sigma(x) - \alpha_i\,\phi_i^2(x)\,\sigma(x)\right]dx = 0$$

(2) $$\therefore \quad \alpha_i = \frac{\int_a^b f(x)\,\phi_i(x)\,\sigma(x)\,dx}{\int_a^b \phi_i^2(x)\,\sigma(x)\,dx} = a_i$$, i.e. α_i are coefficients of generalized Fourier series

Compare (2) with (1)

$$\Rightarrow \boxed{\alpha_i = a_i}$$

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<u>Claim</u> $\alpha_i = a_i$ minimizes the mean-square deviation.

Let us show that the error E is minimized when $\alpha_i = a_i$:

$$E = \int_a^b \Big[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \Big]^2 \sigma(x)\, dx =$$

$$= \int_a^b \Big[f^2 - 2 \sum_{n=1}^{M} \alpha_n f \phi_n + \sum_{n=1}^{M} \sum_{l=1}^{M} \alpha_n \alpha_l \phi_n \phi_l \Big] \sigma\, dx$$

By orthogonality,

$$E = \int_a^b \Big[f^2 - 2 \sum_{n=1}^{M} \alpha_n f \phi_n + \sum_{n=1}^{M} \alpha_n^2 \phi_n^2 \Big] \sigma\, dx =$$

$$= \sum_{n=1}^{M} \Big(\alpha_n^2 \int_a^b \phi_n^2 \sigma\, dx - 2\alpha_n \int_a^b f \phi_n \sigma\, dx \Big) + \int_a^b f^2 \sigma\, dx$$

In $(\ldots)$, factor out $\int_a^b \phi_n^2 \sigma\, dx$ and complete the square.

$(a-b)^2 = a^2 - 2ab + b^2$

$$\alpha_n^2 \int_a^b \phi_n^2 \sigma\, dx - 2\alpha_n \int_a^b f \phi_n \sigma\, dx =$$

$$= \int_a^b \phi_n^2 \sigma\, dx \left[\alpha_n^2 - 2\alpha_n \frac{\int_a^b f \phi_n \sigma\, dx}{\int_a^b \phi_n^2 \sigma\, dx} \right] \underset{\text{square}}{\overset{\text{complete}}{=}}$$

$$= \int_a^b \phi_n^2 \sigma\, dx \left[\underbrace{\alpha_n^2 - 2\alpha_n \frac{\int_a^b f \phi_n \sigma\, dx}{\int_a^b \phi_n^2 \sigma\, dx}}_{a^2 \; - 2a \cdot \; b} + \underbrace{\left(\frac{\int_a^b f \phi_n \sigma\, dx}{\int_a^b f \phi_n \sigma\, dx} \right)^2}_{b^2} - \left(\frac{\int_a^b f \phi_n \sigma\, dx}{\int_a^b f \phi_n \sigma\, dx} \right)^2 \right]$$

$$= \int_a^b \phi_n^2 \sigma\, dx \left(\alpha_n - \frac{\int_a^b f \phi_n \sigma\, dx}{\int_a^b \phi_n^2 \sigma\, dx} \right)^2 - \left(\frac{\int_a^b f \phi_n \sigma\, dx}{\int_a^b f \phi_n \sigma\, dx} \right)^2 \cdot \int_a^b \phi_n^2 \sigma\, dx$$

Hence,

$$E=\sum_{n=1}^{M}\Bigg[\underbrace{\int_a^b \phi_n^2\sigma\,dx}_{>0}\underbrace{\left(\alpha_n-\cancelto{0}{\frac{\int_a^b f\phi_n\sigma\,dx}{\int_a^b \phi_n^2\sigma\,dx}}\right)^2}_{\geq 0}-\underbrace{\left(\frac{\int_a^b f\phi_n\sigma\,dx}{\int_a^b f\phi_n\sigma\,dx}\right)^2}_{\geq 0}\cdot\underbrace{\int_a^b \phi_n^2\sigma\,dx}_{>0}+$$

$$+\underbrace{\int_a^b f^2\sigma\,dx}_{\geq 0} \qquad (3)$$

The only term with α_n is in non-negative form

$\Rightarrow$ E is minimized if

$$\alpha_n-\frac{\int_a^b f\phi_n\sigma\,dx}{\int_a^b \phi_n^2\sigma\,dx}=0$$

or

$$\alpha_n = \frac{\int_a^b f \phi_n \sigma \, dx}{\int_a^b \phi_n^2 \sigma \, dx} = a_n \; : \quad \text{generalized Fourier coefficients}$$

Note the choice $\alpha_n = a_n$ is the "best" choice as it minimizes the mean-square deviation error E.

Note Coefficients $\alpha_n = a_n$ do not depend on M.

Minimum error (see eq^{n} (3)):

$$\boxed{E = \int_a^b f^2 \sigma \, dx - \sum_{n=1}^{M} a_n^2 \int_a^b \phi_n^2 \sigma \, dx}$$

Ex Fourier sine series: $\sigma(x) = 1$, $\quad \phi_n(x) = \sin \frac{n\pi x}{L}$

$$\int_0^L \phi_n^2 \sigma \, dx = \int_0^L \sin^2 \frac{n\pi x}{L} \cdot 1 \, dx = \frac{L}{2}$$

$$\therefore \quad E = \int_0^L f^2 dx - \frac{L}{2} \sum_{n=1}^{M} a_n^2$$

<u>Bessel's inequality</u>:

$$E \geq 0 \quad \Rightarrow \quad \boxed{\int_a^b f^2 \sigma\, dx \geq \sum_{n=1}^{M} a_n^2 \int_a^b \phi_n^2 \sigma\, dx}$$

<u>Parceval's equality</u>:

Since $E = 0$ as $M \to \infty$, we have

$$\boxed{\int_a^b f^2 \sigma\, dx = \sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma\, dx}$$