Recall

Large eigenvalues of Sturm–Liouville problem (asymptotic formula)

Recall Sturm–Liouville equation:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + [q(x) + \lambda\sigma(x)]\,\phi(x) = 0 \qquad a \le x \le b$$

e'values: $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$

$$\lim_{n\to\infty} \lambda_n = \infty$$

λ_1 is the smallest e'value. We can use RQ to approximate small e'values.

It turns out that we can approximate large e'values asymptotically.

Let $\lambda \gg 1$ (λ is very large)

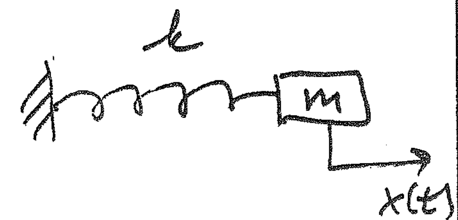
$$\Rightarrow \lambda\sigma(x) \gg q(x)$$

Then S.-L. equation reduces to

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + \lambda\sigma(x)\,\phi(x) = 0$$

$$\underbrace{p(x)}_{m}\frac{d^2\phi}{dx^2} + \underbrace{\frac{dp}{dx}}_{c}\frac{d\phi}{dx} + \underbrace{\lambda\sigma(x)}_{k}\phi = 0 \qquad (1)$$

Think about S.L. equation as of an equation for mass-spring system w/ damping where time is being replaced w/ x and displacement $x(t)$ – with $\phi(x)$



Mass-spring system:

$x(t)$: displacement from equilibrium
m: mass
k: spring constant
c: damping coefficient

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$$m\ddot{x} + c\dot{x} + kx = 0$$

The restoring force $F(x) = -kx$. In our case, this force is $-\lambda\sigma(x)\phi(x)$. Since λ is large, the restoring force is also large $\Rightarrow$ $\phi(x)$ is rapidly oscillating function. This is in agreement with the fact that $\phi(x)$ has many zeros for large λ.

Since $\phi(x)$ is rapidly oscillating function, functions $p(x)$ and $\sigma(x)$ are slowly varying functions compared to $\phi(x)$.

$$\therefore \quad p(x) \approx p(x_0) \qquad \sigma(x) \approx \sigma(x_0)$$

$$\Downarrow$$

$$\frac{dp}{dx} \approx 0$$

for all x near any pt x_0. Hence, eq^n (1) can be written as (approximated)

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$$\underbrace{p(x_0)}_{\substack{\text{const} \\ m}} \frac{d^2\phi}{dx^2} + \lambda \underbrace{\sigma(x_0)}_{\substack{\text{const} \\ k}} \phi(x) = 0 \qquad \text{near any pt } x_0$$

Recall solution to a mass-spring system eq^n:

$$m\ddot{x} + kx = 0 \qquad\qquad x = e^{rx}$$

$$\ddot{x} + \frac{k}{m}x = 0 \qquad\qquad \text{let } \omega_0^2 = \frac{k}{m}$$

Characteristic equation: $r^2 + \frac{k}{m} = 0$

Then $\omega_0 = \sqrt{\frac{k}{m}}$ and $r = \pm i\omega_0$

ω_0: angular frequency / local frequency

$$x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$$

or $x(t) = A \sin(\omega_0 t - \alpha)$ (2)

where $A = \sqrt{C_1^2 + C_2^2}$: amplitude

α: phase shift

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In our case, $m = p(x_0)$, $k = 2\sigma(x_0)$. Then the angular/local frequency can be written as

$$\text{"}\omega_0\text{"} = \begin{matrix}\text{local} \\ \text{frequency}\end{matrix} = \sqrt{\frac{2\sigma(x_0)}{p(x_0)}} \Rightarrow \boxed{\phi(x) \approx A \sin\left(\sqrt{\frac{2\sigma(x_0)}{p(x_0)}}\, x - \alpha\right)}$$

If we use this expression for frequency, then $\phi(x)$ will be an oscillatory function w/ constant amplitude like in (2).

a better approximation to $\phi(x)$ will be if we assume that

$$\phi(x) = A(x) \sin(\psi(x))$$

where

$A(x)$: slowly varying amplitude

$\psi(x)$: phase

Expand $\psi(x)$ in a Taylor series about $x = x_0$:

$$\psi(x) = \psi(x_0) + \psi'(x_0)(x - x_0) + \dots$$

$$\Rightarrow \sin(\psi(x)) = \sin(\underline{\underline{\psi'(x_0)}}(x - x_0) + \psi(x_0) + \dots)$$

compare this with $\sin(\underline{\underline{\omega_0}} t - \alpha)$

$\therefore$ $\phi(x)$ oscillates w/ local frequency given by

$$\psi'(x_0) = \sqrt{\frac{\lambda\sigma(x_0)}{p(x_0)}} \qquad a \le x \le b$$

Then $$\psi(x) = \int_a^x \sqrt{\frac{\lambda\sigma(s)}{p(s)}}\, ds + \psi(a) \quad : \text{exact}$$

Check: $$\psi'(x) = \sqrt{\frac{\lambda\sigma(x)}{p(x)}}$$

$$\psi(a) = \psi(a) \quad \checkmark$$

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For large λ,

$$\psi(x) \approx \int_a^x \left(\frac{\lambda\,\sigma(s)}{p(s)}\right)^{1/2} ds$$

It can be shown that

$$A(x) = [\sigma(x)p(x)]^{-1/4} \qquad \text{(to the leading order)}$$

$$\therefore \quad \boxed{\phi(x) = [\sigma(x)p(x)]^{-1/4} \sin \int_a^x \left[\frac{\lambda\,\sigma(s)}{p(s)}\right]^{1/2} ds}$$

approximation of an e'function $\phi(x)$ for large λ

To find λ, we need to use BCs.

For example,

$$\phi(0) = \phi(L) = 0$$

$$0 = \phi(L) = \left[\sigma(L)p(L)\right]^{-1/4} \sin \int_0^L \left[\frac{\lambda\sigma(s)}{p(s)}\right]^{1/2} ds$$

$$a = 0$$

$$\therefore \quad \sin \int_0^L \left[\frac{\lambda\sigma(s)}{p(s)}\right]^{1/2} ds = 0$$

$$\int_0^L \left[\frac{\lambda\sigma(s)}{p(s)}\right]^{1/2} ds = n\pi, \qquad n = 1, 2, \ldots$$

$$\lambda^{1/2} \int_0^L \left[\frac{\sigma(s)}{p(s)}\right]^{1/2} ds = n\pi, \qquad n = 1, 2, \ldots$$

$$\therefore \quad \boxed{\lambda = \frac{(n\pi)^2}{\left\{\int_0^L \left[\frac{\sigma(s)}{p(s)}\right]^{1/2} ds\right\}^2}}$$

approximation of large e'values λ_n

Ex $\phi'' + \lambda(1+x)\phi = 0$ $\qquad p = 1, \quad q(x) = 0, \quad \sigma(x) = 1+x$

$\phi(0) = \phi(1) = 0$

Hence,

$$\lambda \approx \frac{n^2\pi^2}{\left[\int_0^1 \left(\frac{1+s}{1}\right)^{1/2} ds\right]^2} = \frac{n^2\pi^2}{\frac{2}{3}(1+s)^{3/2}\Big|_0^1} = \boxed{\frac{n^2\pi^2}{\frac{4}{9}\left(2^{3/2}-1\right)^2}}$$

gives very good approximation for small e'values

TABLE 5.9.1: Eigenvalues λ_n

n	Numerical answer* (assumed accurate)	Asymptotic formula (5.9.11)	Error	% error
1	6.548395	6.642429	0.094034	1.436
2	26.464937	26.569718	0.104781	0.3950
3	59.674174	59.781865	0.107691	0.1805
4	106.170023	106.278872	0.108849	0.1025
5	165.951321	166.060737	0.109416	0.0659
6	239.0177275	239.1274615	0.109734	0.0459
7	325.369115	325.479045	0.109930	0.0338

*Courtesy of E. C. Gartland, Jr.

ERCISES 5.9

5.9.1. Estimate (to leading order) the large eigenvalues and corresponding eigenfunctions for

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + [\lambda\sigma(x) + q(x)]\phi = 0$$

if the boundary conditions are

(a) $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

*(b) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

(c) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) + h\phi(L) = 0$

5.9.2. Consider

$$\frac{d^2\phi}{dx^2} + \lambda(1+x)\phi = 0$$

subject to $\phi(0) = 0$ and $\phi(1) = 0$. Roughly sketch the eigenfunctions for λ large. Take into account amplitude and period variations.

5.9.3. Consider for $\lambda \gg 1$

$$\frac{d^2\phi}{dx^2} + [\lambda\sigma(x) + q(x)]\phi = 0.$$

*(a) Substitute

Eigenvalues λ_n: comparison between numerically and asymptotically computed eigenvalues

λ_n^{numer}	λ_n^{asympt}	absolute error= $= \lvert\lambda_n^{numer} - \lambda_n^{asympt}\rvert$	relative error= $= \frac{\lvert\lambda_n^{numer} - \lambda_n^{asympt}\rvert}{\lambda_n^{numer}}$
6.548395	6.642429	0.094034	0.014360
26.464937	26.569718	0.104781	0.003959
59.674174	59.781865	0.107691	0.001805
106.170023	106.278872	0.108849	0.001025
165.951321	166.060737	0.109416	0.000659
239.017728	239.127462	0.109734	0.000459
325.369115	325.479045	0.109930	0.000338

Note: The relative error (percent error) decreases as n increases where the absolute error stays about the same. This approximately constant error can be improved by adding a correction term to the asymptotic formula for eigenvalues.

Approximation Properties

We know that any piecewise smooth function can be represented by a generalized Fourier series:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$

By orthogonality with weight $\sigma(x)$:

$$(1) \quad a_n = \frac{\int_a^b f(x)\,\phi_n(x)\,\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\,\sigma(x)\,dx} \; : \; \text{generalized Fourier coefficients}$$

What if we write

$$f(x) \sim \sum_{n=1}^{M} \alpha_n \phi_n(x) \qquad a \le x \le b$$

i.e. we only use the first M e'functions $\phi_n(x)$.

<u>Q</u> Which choice of α_n's would give the "best" result, i.e. the "best" approximation of $f(x)$?

<u>Q</u> Should α_n's ~~depend~~ on M?

<u>Q</u> What is the error?

We need to define what the "best" approximation is.

Best approximation = least error

We could use

$$\max_{a \le x \le b} \Big| \underbrace{f(x)}_{\text{exact}} - \underbrace{\sum_{n=1}^{M} \alpha_n \Phi_n(x)}_{\text{approximation}} \Big| = \text{error}$$

Not convenient to use.

Instead, we will use <u>Mean-Square Deviation</u>:

$$E \overset{\text{def}}{=} \int_a^b \Big[f(x) - \sum_{n=1}^{M} \alpha_n \Phi_n(x) \Big]^2 \underbrace{\sigma(x)}_{>0} \, dx$$

Note: $E \geq 0$ and $E = 0$ iff $f(x) = \sum_{n=1}^{M} \alpha_n \phi_n(x)$

Think of E as of a function of $\alpha_1, \alpha_2, \ldots, \alpha_M$:

$$E = E(\alpha_1, \alpha_2, \ldots, \alpha_M).$$

Q How do we find critical points of E?

$$\frac{\partial E}{\partial \alpha_i} = 0 \qquad i = 1, \ldots, M$$

$$0 = \frac{\partial E}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i} \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right]^2 \sigma(x)\, dx =$$

$$= \int_a^b 2\left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right] \cdot \left(-\phi_i(x) \right) \cdot \sigma(x)\, dx =$$

Orthogonality: $\displaystyle\int_a^b \phi_n(x)\,\phi_i(x)\,\sigma(x)\,dx = 0 \quad \text{if } i \neq n$

$$= -2\int_a^b \left[f(x)\,\phi_i(x)\,\sigma(x) - \alpha_i\,\phi_i^2(x)\,\sigma(x) \right] dx = 0$$

(2) $$\therefore \quad \alpha_i = \frac{\displaystyle\int_a^b f(x)\,\phi_i(x)\,\sigma(x)\,dx}{\displaystyle\int_a^b \phi_i^2(x)\,\sigma(x)\,dx} = a_i ,$$

i.e. α_i are generalized Fourier coefficients

Compare (2) with (1).

$$\therefore \quad \boxed{\alpha_i = a_i}$$

Claim : $\alpha_i = a_i$ minimizes the mean-square deviation

Let us show that the error E is minimized when $\alpha_i = a_i$.

$$E = \int_a^b \Big[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \Big]^2 \sigma(x)\, dx =$$

$$= \int_a^b \Big[f^2 - 2 \sum_{n=1}^{M} \alpha_n f \phi_n + \sum_{n=1}^{M} \sum_{\ell=1}^{M} \alpha_n \alpha_\ell \phi_n \phi_\ell \Big] \sigma(x)\, dx$$

By orthogonality,

$$E = \int_a^b \Big[f^2 - 2 \sum_{n=1}^{M} \alpha_n f \phi_n + \sum_{n=1}^{M} \alpha_n^2 \phi_n^2 \Big] \sigma(x)\, dx =$$

$$= \sum_{n=1}^{M} \left(\alpha_n^2 \int_a^b \phi_n^2 \sigma \, dx - 2\alpha_n \int_a^b f \phi_n \sigma \, dx \right) + \int_a^b f^2 \sigma \, dx$$

In $(\ldots)$, factor out $\int_a^b \phi_n^2 \sigma(x) \, dx$ and complete the square

$$\alpha_n^2 \int_a^b \phi_n^2 \sigma \, dx - 2\alpha_n \int_a^b f \phi_n \sigma \, dx =$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$= \int_a^b \phi_n^2 \sigma \, dx \left[\underbrace{\alpha_n^2}_{a^2} - \underbrace{2\alpha_n \frac{\int_a^b f \phi_n \sigma \, dx}{\int_a^b \phi_n^2 \sigma \, dx}}_{2 \cdot a \cdot b} \right] \underset{\text{square}}{\overset{\text{complete}}{=}}$$

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$$= \int_a^b \phi_n^2 \sigma\, dx \left[\alpha_n^2 - 2\alpha_n \underbrace{\frac{\int_a^b f \phi_n \sigma\, dx}{\int_a^b \phi_n^2 \sigma\, dx}}_{b} + \left(\frac{\int_a^b f \phi_n \sigma\, dx}{\int_a^b \phi_n^2 \sigma\, dx} \right)^2 - \right.$$

$$\left. - \left(\frac{\int_a^b f \phi_n \sigma\, dx}{\int_a^b \phi_n^2 \sigma\, dx} \right)^2 \right] =$$

$$= \int_a^b \phi_n^2 \sigma\, dx \left(\alpha_n - \frac{\int_a^b f \phi_n \sigma\, dx}{\int_a^b \phi_n^2 \sigma\, dx} \right)^2 - \left(\frac{\int_a^b f \phi_n \sigma\, dx}{\int_a^b \phi_n^2 \sigma\, dx} \right) \cdot \int_a^b \phi_n^2 \sigma\, dx$$

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Hence,

$$E=\sum_{n=1}^{M}\Bigg[\overbrace{\int_a^b \phi_n^2\sigma\,dx}^{>0}\underbrace{\left(\alpha_n-\frac{\int_a^b f\phi_n\sigma\,dx}{\int_a^b \phi_n^2\sigma\,dx}\right)^2}_{\geq 0} - \underbrace{\left(\frac{\int_a^b f\phi_n\sigma\,dx}{\int_a^b \phi_n^2\sigma\,dx}\right)^2}_{\geq 0}\cdot\underbrace{\int_a^b \phi_n^2\sigma\,dx}_{>0} + \underbrace{\int_a^b f^2\sigma\,dx}_{\geq 0}$$

The only term with α_n is in non-negative form,

$\Rightarrow$ E is minimized when

$$\alpha_n-\frac{\int_a^b f\phi_n\sigma\,dx}{\int_a^b \phi_n^2\sigma\,dx}\to 0 \quad \text{i.e. } \alpha_n=a_n$$