Lecture 38

# Higher Dimensional PDEs (Ch 7)

Review PDEs w/ 2 independent variables:

Heat equation: $u_t = k u_{xx}$ $(x, t)$

Laplace equation: $u_{xx} + u_{yy} = 0$ $(x, y)$

Wave equation: $u_{tt} = c^2 u_{xx}$ $(x, t)$

Next we consider higher dimensional PDEs:

Heat equation: $u_t = k(u_{xx} + u_{yy})$ $(x, y, t)$

or $u_t = k(u_{xx} + u_{yy} + u_{zz})$ $(x, y, z, t)$

Laplace equation: $u_{xx} + u_{yy} + u_{zz} = 0$ $(x, y, z)$

Wave equation:

$$u_{tt} = c^2(u_{xx} + u_{yy}) \qquad (x, y, t)$$

or

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}) \qquad (x, y, z, t)$$

or we can write RHS as

$$u_{tt} = c^2 \Delta u$$

where

$$\Delta u = u_{xx} + u_{yy} \quad \text{in 2D}$$

$$\Delta u = u_{xx} + u_{yy} + u_{zz} \quad \text{in 3D}$$

The choice of coordinate system depends on the geometry. For rectangular domain, use Cartesian or rectangular coordinates.

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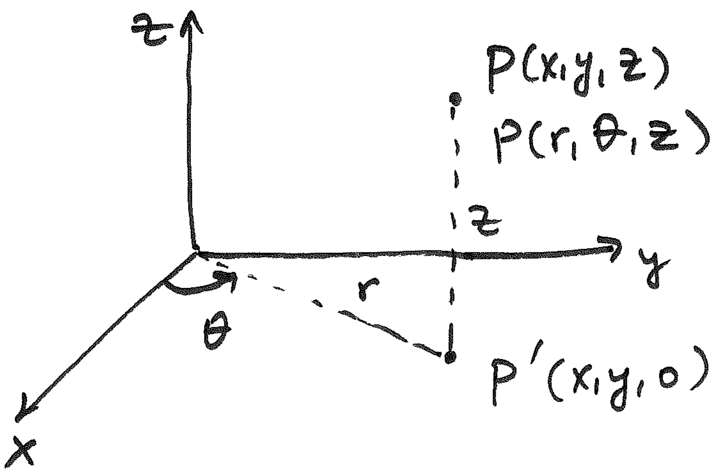
If a domain has symmetry wrt an axis, choose cylindrical coordinate system and allign the axis of symmetry with $z$-axis. Examples: cylinders and cones.




$x = r\cos\theta$ $\qquad r \geq 0$

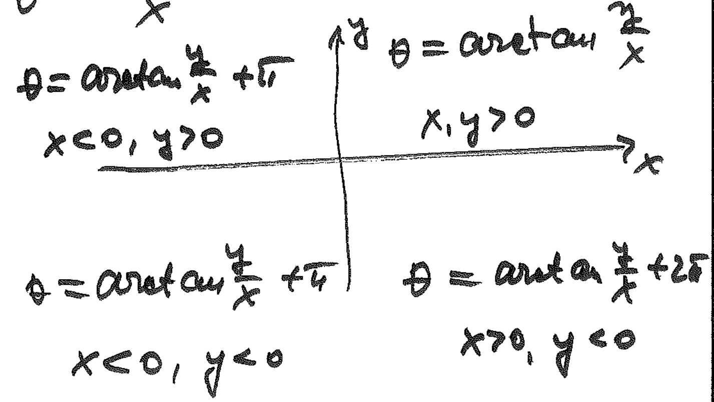
$y = r\sin\theta$ $\qquad 0 \leq \theta \leq 2\pi$

$z = z$

$r^2 = x^2 + y^2$

$r = \sqrt{x^2+y^2}$ $\qquad z = z$

$\tan\theta = \frac{y}{x}$

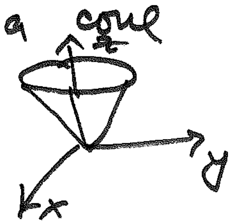



Ex $z = r \geq 0$

$r = \sqrt{x^2+y^2} \Rightarrow z = \sqrt{x^2+y^2}$

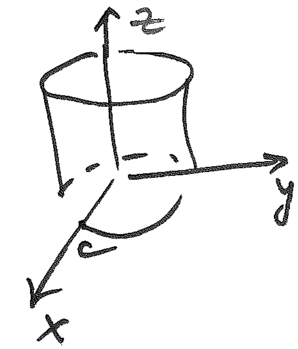
or $z^2 = x^2 + y^2$ : cone

since $z \geq 0 \Rightarrow$ upper half of a cone

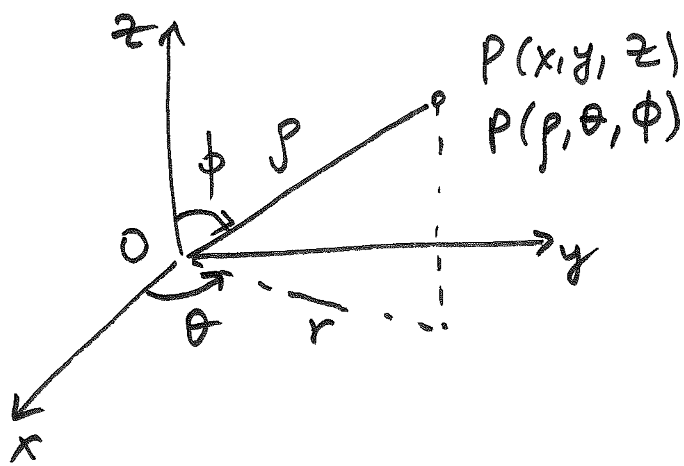


Ex $r = \text{const}$, ie. $r = c$ $\Rightarrow$ $r^2 = c^2$

$x^2+y^2=c^2$ : cylinder




If a region is symmetric wrt a point, it is convenient to use spherical coordinates $(\rho, \theta, \phi)$.




$\rho \geq 0$ $\quad 0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

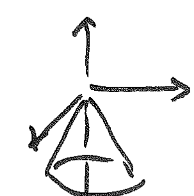
$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$\rho^2 = x^2+y^2+z^2$ $\quad \tan\theta = \frac{y}{x}$

$\cos\phi = \frac{z}{\rho}$

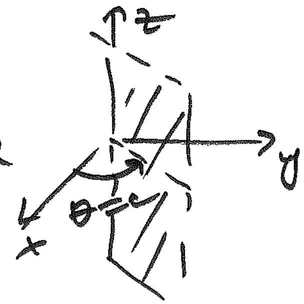



Ex $\phi = c$ : cone

$0 < c < \frac{\pi}{2}$ : upper half cone; $\quad \frac{\pi}{2} < c < \pi$ : lower half of cone

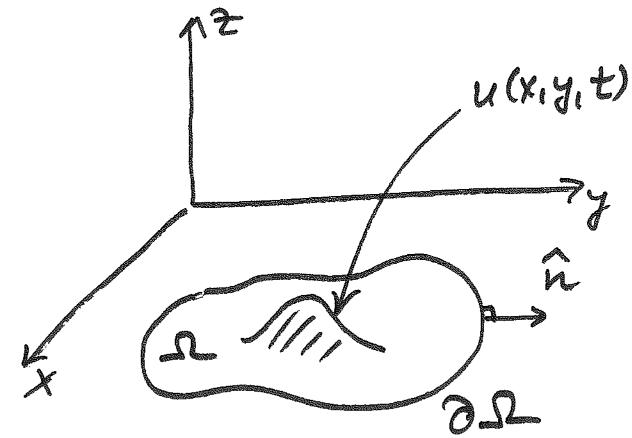
Ex $\rho = c$ : sphere centered at origin w/ rad. c

Ex $\theta = c$ : vertical plane




## Separation of time variable

Ex Vibrating membrane (2D + time)




$u_{tt} = c^2(u_{xx} + u_{yy})$ $\quad (x,y) \in \Omega$

$$\left.\begin{aligned} u(x,y,0) &= \alpha(x,y) \\ u_t(x,y,0) &= \beta(x,y) \end{aligned}\right\} \text{ICs} \qquad (x,y) \in \Omega$$

BC: $\beta_1 u + \beta_2 \nabla u \cdot \hat{n} = 0$ on $\partial\Omega$ : elastic BC

$\hat{n}$ : outward unit normal vector to $\partial\Omega$

$\hat{n} = \langle n_1, n_2 \rangle$ $\qquad \nabla u = \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle$

$\beta_1, \beta_2$: const

$\beta_2 = 0$ : Dirichlet BC

$\beta_1 = 0$ : Neumann BC

Separation of variables:

$$u(x,y,t) = h(t)\,\phi(x,y)$$

$$\frac{d^2 h}{dt^2}\phi = c^2 h\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) \quad \Big| \; \frac{1}{h\phi c^2}$$

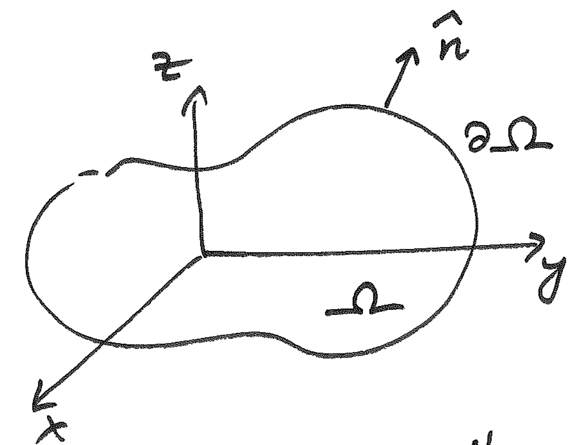
$$\frac{d^2h/dt^2}{c^2 h} = \frac{\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2}{\phi} = -\lambda$$

$$\boxed{\begin{aligned} &\frac{d^2 h}{dt^2} + c^2\lambda h = 0 \\ &\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \lambda\phi = 0 \end{aligned}}$$

BCs: $\beta_1 \Phi + \beta_2 \nabla \Phi \cdot \hat{n} = 0$ on $\partial\Omega$

Note usually time-dependent part is easier to solve and spatial part requires more work.

<u>Ex</u> Heat equation (3D + time)




$$u_t = k(u_{xx} + u_{yy} + u_{zz})$$

IC: $u(x,y,z,0) = \alpha(x,y,z)$, $(x,y,z) \in \Omega$

$\hat{n}$: outward unit normal

$\hat{n} = \langle n_1, n_2, n_3 \rangle$

BC: $\beta_1 u + \beta_2 \nabla u \cdot \hat{n} = 0$ on $\partial\Omega$

$\nabla u = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle$

$\Omega$: domain in 3D

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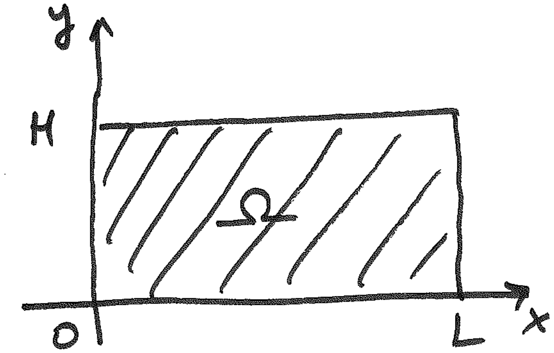
Separation of variables:

$$u(x,y,z,t) = h(t)\,\phi(x,y,z)$$

$$\boxed{\begin{aligned} &\frac{dh}{dt} + \lambda k h = 0 \\ &\phi_{xx} + \phi_{yy} + \phi_{zz} = -\lambda \phi \quad \text{in } \Omega \\ &\text{BC: } \beta_1 \phi + \beta_2 \nabla \phi \cdot \hat{n} = 0 \quad \text{on } \partial\Omega \end{aligned}}$$

**Ex** Vibrating Rectangular Membrane




$$u_{tt} = c^2 (u_{xx} + u_{yy})$$

$$\Omega = \{(x,y): \; 0 \le x \le L, \; 0 \le y \le H\}$$

$$u(x,y,0) = \alpha(x,y)$$

$$u_t(x,y,0) = \beta(x,y) \qquad (x,y) \in \Omega$$

BCs : $u=0$ on $\partial\Omega \iff u(0,y,t) = u(L,y,t) = 0$

$u(x,0,t) = u(x,H,t) = 0$

Separation of variables :

$$u(x,y,t) = h(t)\,\phi(x,y)$$

h-equation

$$\frac{d^2h}{dt^2} + \lambda c^2 h = 0$$

$$h(t) = C_1 \cos(\sqrt{\lambda}\,c\,t) + C_2 \sin(\sqrt{\lambda}\,c\,t)$$

$\phi$- equation

$$\phi_{xx} + \phi_{yy} = -\lambda\phi$$

$$\phi(0,y) = \phi(L,y) = 0$$
$$\phi(x,0) = \phi(x,H) = 0$$

Another separation of variables:

$$\phi(x,y) = f(x)\, g(y)$$

$$\frac{d^2 f}{dx^2} \cdot g(y) + f(x) \frac{d^2 g}{dy^2} = -\lambda\, f(x)\, g(y) \qquad \Big/ \ \frac{1}{fg}$$

$$\frac{d^2 f/dx^2}{f(x)} + \frac{d^2 g/dy^2}{g(y)} = -\lambda$$

$$\underbrace{\frac{d^2 f/dx^2}{f(x)}}_{f^n \text{ of } x} = \underbrace{-\frac{d^2 g/dy^2}{g(y)} - \lambda}_{f^n \text{ of } y} = -\mu$$

$$\boxed{\begin{array}{lll} f'' + \mu f = 0 & \text{w/} & f(0) = f(L) = 0 \\ \ddot{g} + (\lambda - \mu) g = 0 & \text{w/} & g(0) = g(H) = 0 \end{array}}$$

$$'' = \frac{d^2}{dx^2}$$

$$\cdot\cdot = \frac{d^2}{dy^2}$$

$\Rightarrow f_n(x) = \sin\frac{n\pi x}{L}$ $\qquad \mu_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \ldots$

$g_m(y) = \sin\frac{m\pi y}{H}$ $\qquad \lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \quad m = 1, 2, 3 \ldots$

$\therefore \quad \lambda_{nm} = \mu_n + \left(\frac{m\pi}{H}\right)^2$

e'values: $\boxed{\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \qquad n, m = 1, 2, \ldots}$

e'functions: $\boxed{\phi_{nm}(x, y) = \sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi y}{H}, \quad n, m = 1, 2, \ldots}$