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Lecture 39

Last time we found

$$\text{e'values:} \quad \lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \quad n,m = 1,2,\ldots$$

$$\text{e'functions:} \quad \phi_{nm}(x,y) = \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}, \quad n,m = 1,2,\ldots$$

Recall

$h(t) = C_1 \cos(\sqrt{\lambda}\, ct) + C_2 \sin(\sqrt{\lambda}\, ct)$

By principle of superposition,

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \cos(\sqrt{\lambda_{nm}}\, ct) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \sin(\sqrt{\lambda_{nm}}\, ct)$$

Next we use ICs and orthogonality to find A_{nm} and B_{nm}.

$$\alpha(x,y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \qquad \Big| \cdot \sin\frac{\tilde{m}\pi y}{H} \quad \int_0^H$$

Orthogonality I:

$$\int_0^H \alpha(x,y)\cdot \sin\frac{\tilde{m}\pi y}{H}\,dy = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L} \underbrace{\int_0^H \sin\frac{m\pi y}{H} \sin\frac{\tilde{m}\pi y}{H}\,dy}_{= \begin{cases} 0, & m \neq \tilde{m} \\ \frac{H}{2}, & m = \tilde{m} \end{cases}}$$

Change $\tilde{m} \to m$:

$$\frac{2}{H}\int_0^H \alpha(x,y) \sin\frac{m\pi y}{H}\,dy = \sum_{n=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L} \qquad \Big| \quad \sin\frac{\tilde{n}\pi x}{L} \qquad \int_0^L$$

Orthogonality II:

$$\frac{2}{H}\int_0^L\int_0^H \alpha(x,y)\sin\frac{m\pi y}{H}\cdot\sin\frac{\tilde{n}\pi x}{L}\,dy\,dx = \sum_{n=1}^{\infty} A_{nm}\underbrace{\int_0^L \sin\frac{n\pi x}{L}\sin\frac{\tilde{n}\pi x}{L}\,dx}_{\begin{cases}0, & n\neq\tilde{n}\\ \frac{L}{2}, & n=\tilde{n}\end{cases}}$$

$\tilde{n}\to n$

$$\therefore\ \boxed{A_{nm}=\frac{4}{LH}\int_{x=0}^{L}\int_{y=0}^{H}\alpha(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dy\,dx\ ,\quad n,m=1,2,\dots}$$

$$\beta(x,y)=u_t(x,y,0)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\underbrace{B_{nm}\cdot\sqrt{\lambda_{nm}}\cdot c}_{\text{like } A_{nm}}\cdot\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$$

Hence,

$$B_{nm} \cdot \sqrt{\lambda_{nm}} \cdot c = \frac{4}{LH} \int_{x=0}^{L} \int_{y=0}^{H} \beta(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dy\, dx$$

or

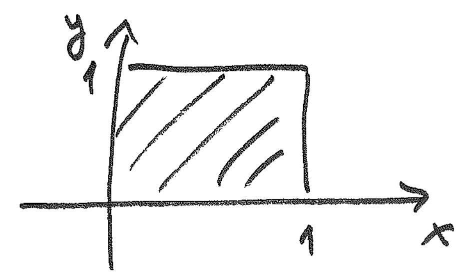
$$B_{nm} = \frac{1}{c\sqrt{\lambda_{nm}}} \cdot \frac{4}{LH} \int_{0}^{L} \int_{0}^{H} \beta(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dy\, dx$$

where

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \qquad n, m = 1, 2, \ldots$$

Ex Consider the case when $L = 1$, $H = 1$. Set $c = 1$.

Initially flat membrane $\Rightarrow \alpha(x,y) = 0 \Rightarrow A_{nm} = 0$



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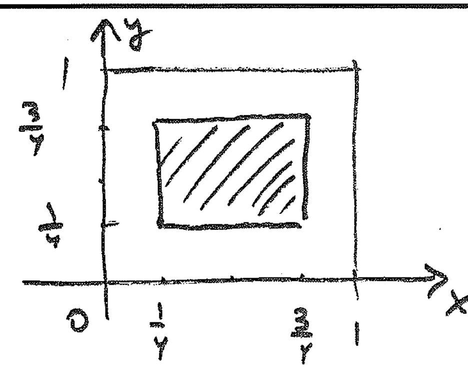
Initial "kick":

$$\beta(x,y) = \begin{cases} 1, & \frac{1}{4} \le x, y \le \frac{3}{4} \\ 0, & \text{otherwise} \end{cases}$$



$$B_{nm} = \frac{4}{\sqrt{\lambda_{nm}}} \int_{x=\frac{1}{4}}^{\frac{3}{4}} \int_{y=\frac{1}{4}}^{3/4} \sin\frac{m\pi y}{1} \sin\frac{n\pi x}{1}\, dy\, dx =$$

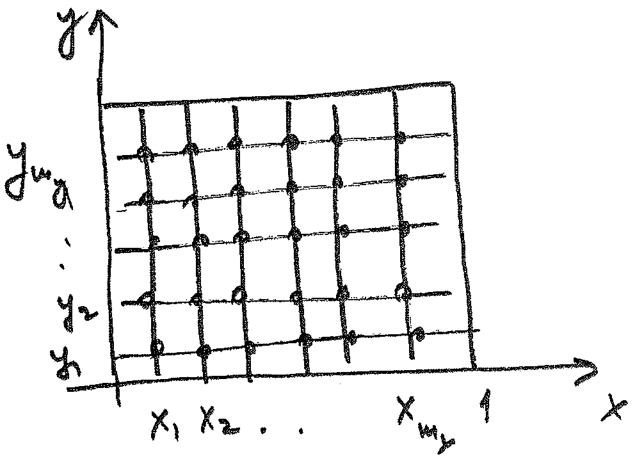
$$= \frac{4}{\sqrt{\lambda_{nm}}} \int_{x=\frac{1}{4}}^{3/4} \sin n\pi x\, dx \cdot \int_{y=\frac{1}{4}}^{3/4} \sin m\pi y\, dy =$$

$$= \frac{4}{\sqrt{\lambda_{nm}}} \cdot \frac{1}{nm\pi^2} \left(\cos\frac{3n\pi}{4} - \cos\frac{n\pi}{4}\right)\left(\cos\frac{3m\pi}{4} - \cos\frac{m\pi}{4}\right)$$

$$\lambda_{nm} = (n\pi)^2 + (m\pi)^2$$

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \sin n\pi x \sin m\pi y \cdot \sin(\sqrt{\lambda_{nm}}\, t)$$

$$u(x,y,t) \approx \sum_{n=1}^{N} \sum_{m=1}^{M} B_{nm} \sin n\pi x \cdot \sin m\pi y \cdot \sin(\sqrt{\lambda_{nm}}\, t)$$



m_x points in x-direction

m_y points in y-direction

Let's compute solution at

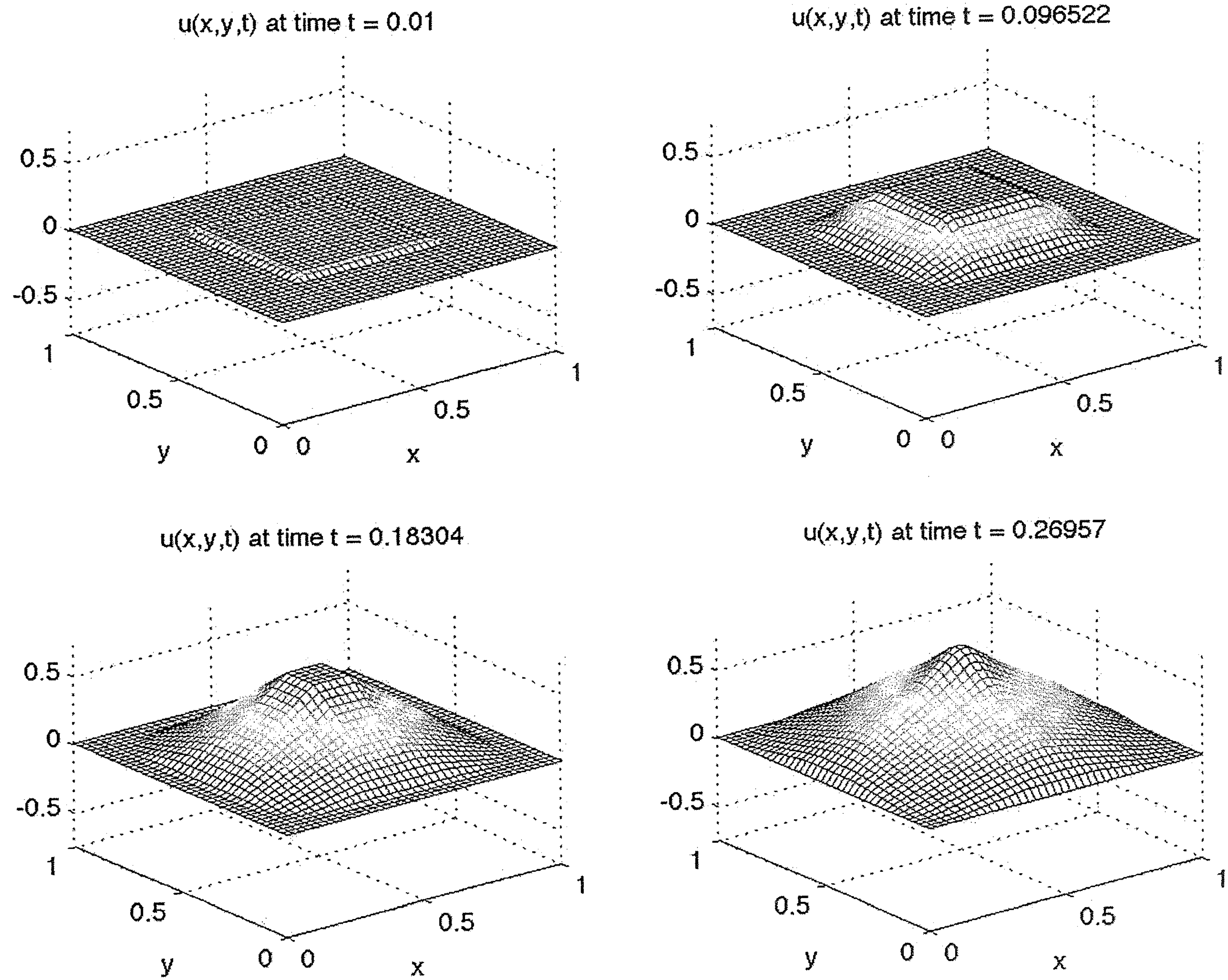
$m_t = 12$ instances of time

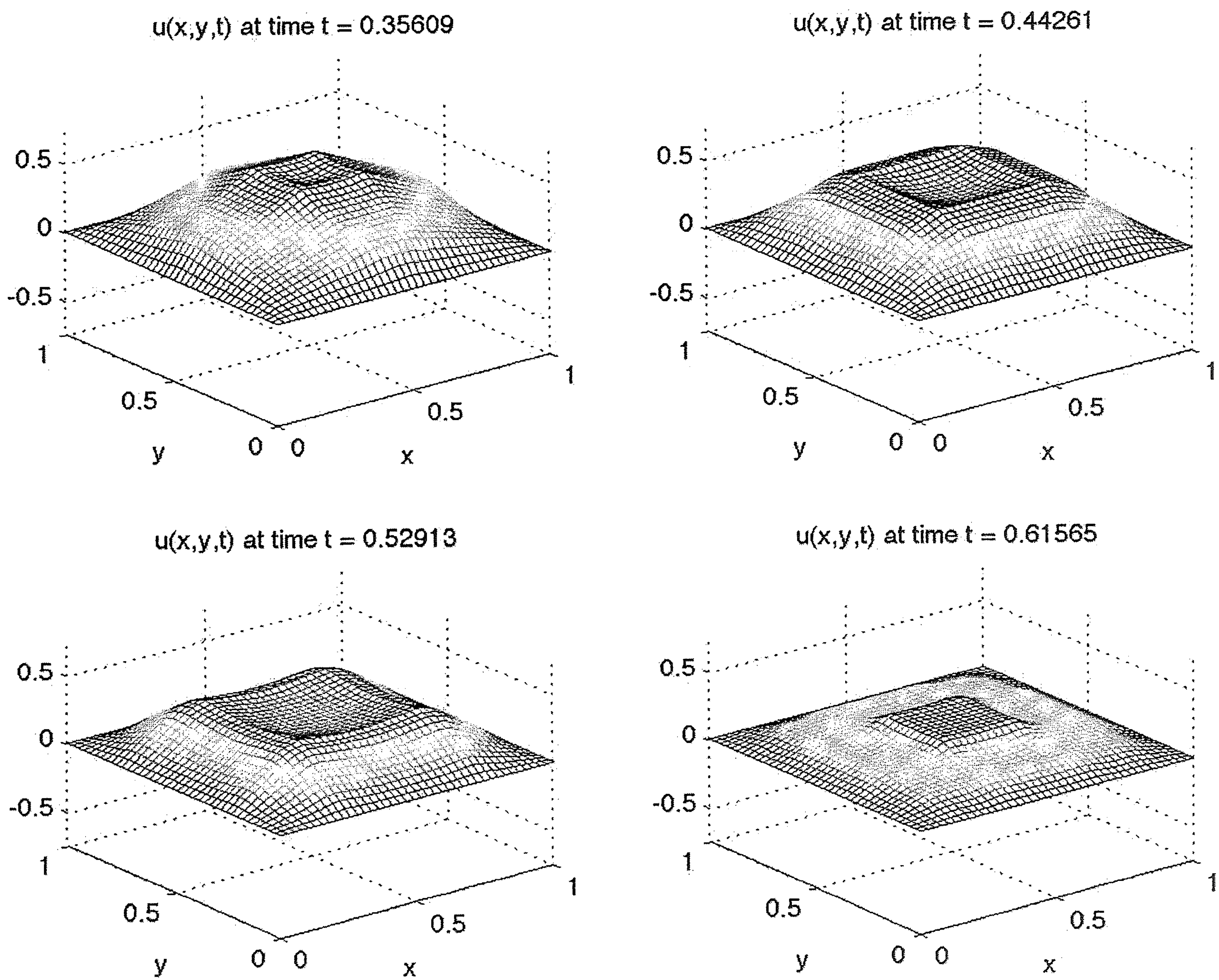
Let $N = M = 50$, $m_x = m_y = 40$, $m_t = 12$

2D $50^2 \cdot 40^2 \cdot 12 = 48$ million

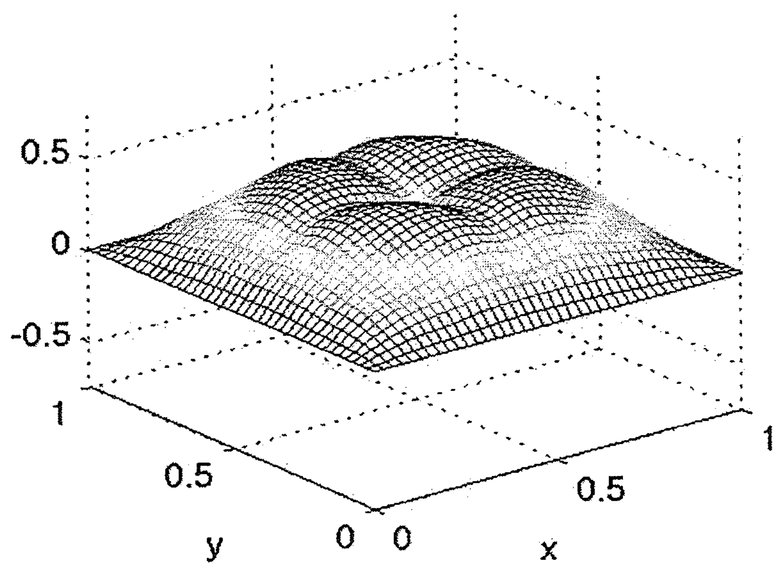
1D $50 \cdot 40 \cdot 12 = 24{,}000$

See these graphs and a Matlab code on the course webpage

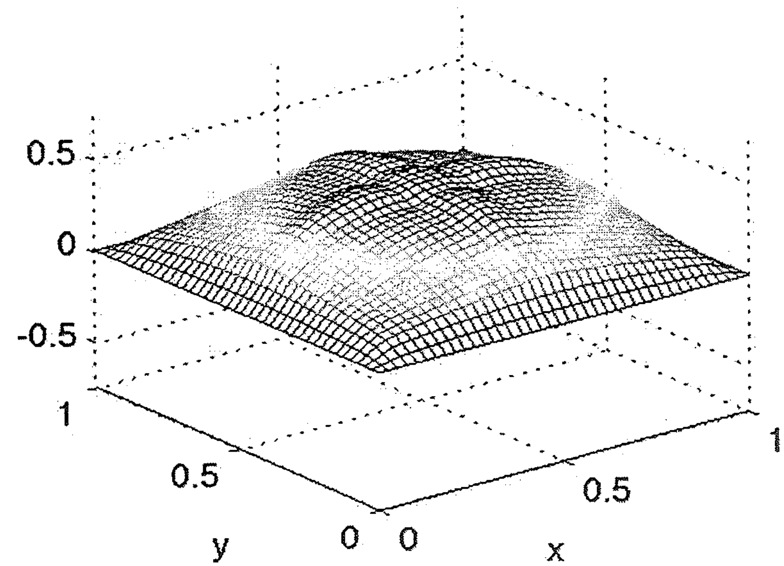



u(x,y,t) at time t = 0.35609
u(x,y,t) at time t = 0.44261
u(x,y,t) at time t = 0.52913
u(x,y,t) at time t = 0.61565
0.5
0
-0.5
1
0.5
0
y
0
0.5
1
x

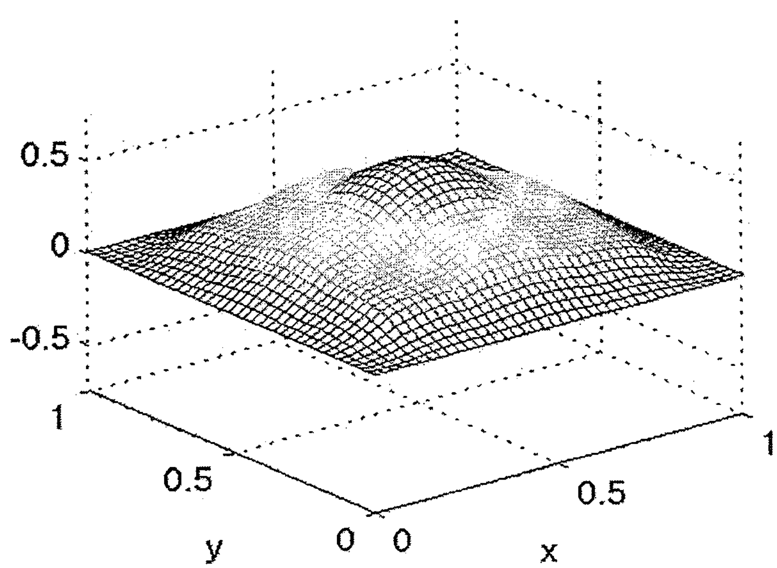
u(x,y,t) at time t = 1.7404



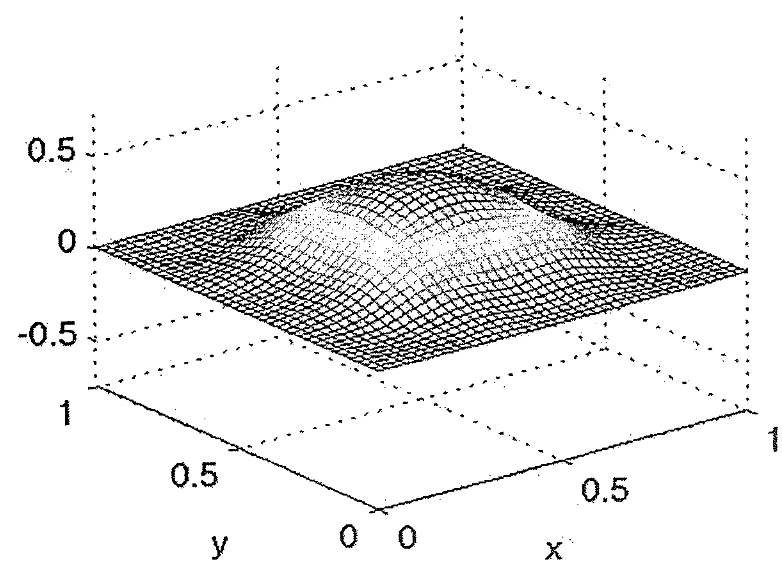
u(x,y,t) at time t = 1.827

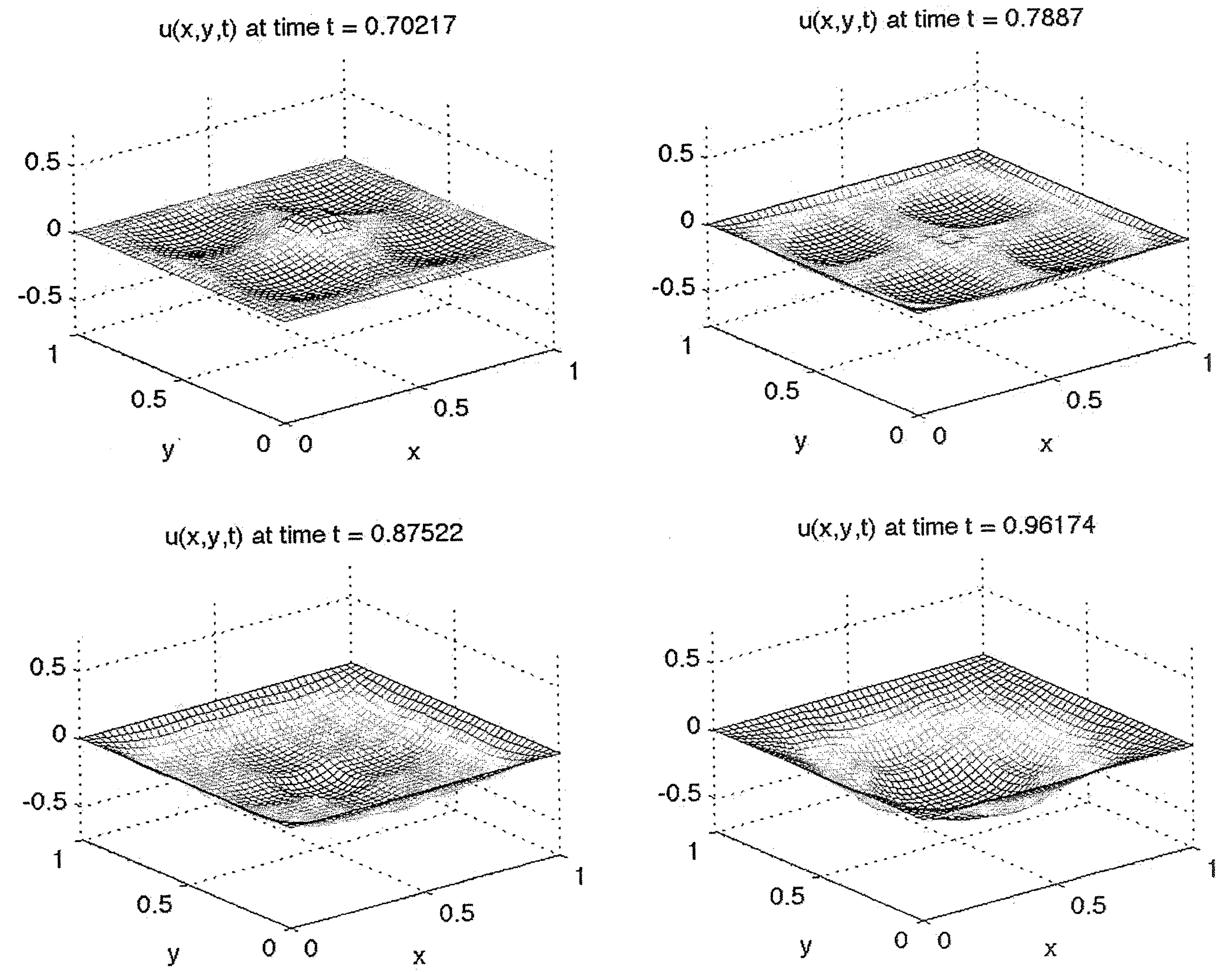


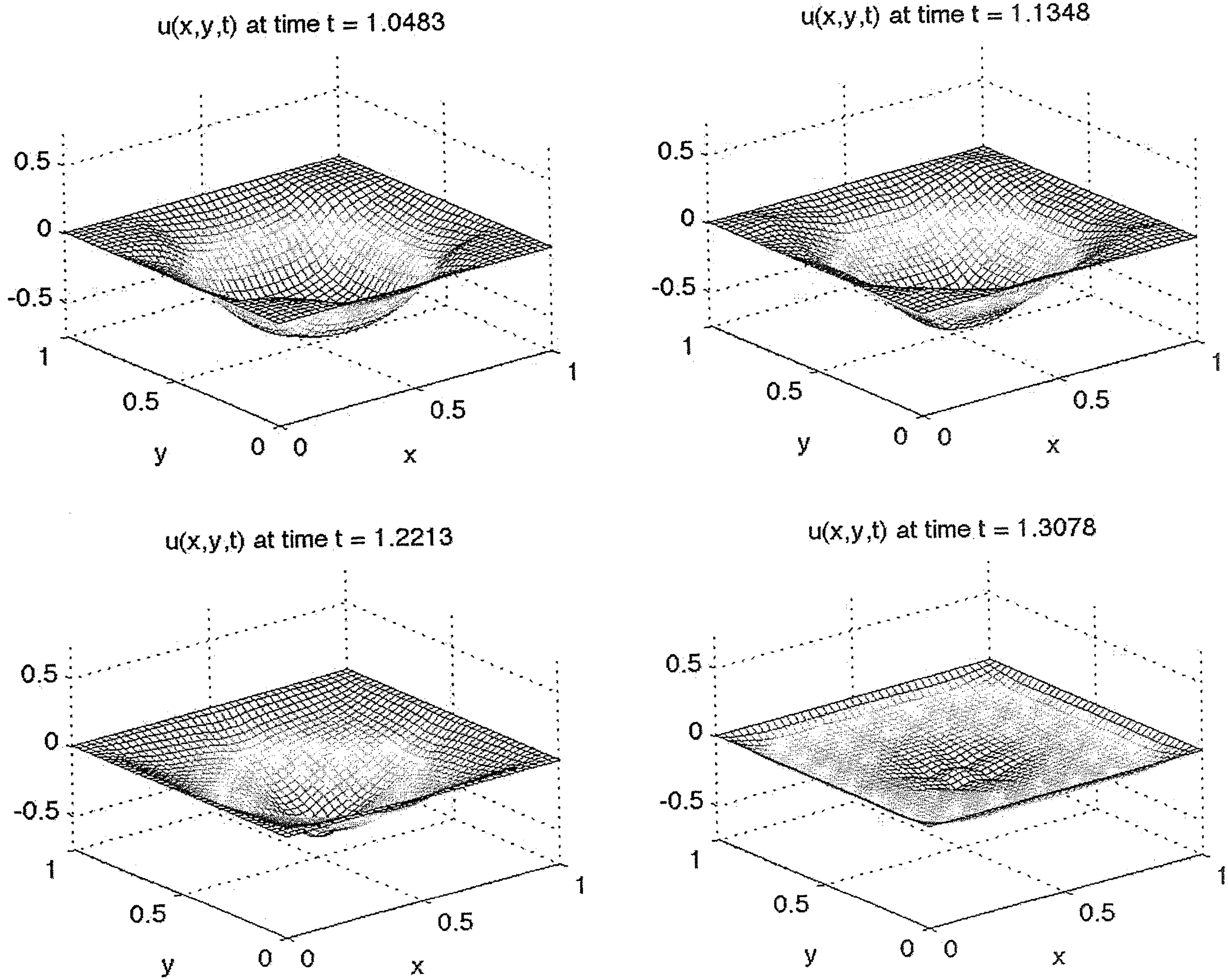
u(x,y,t) at time t = 1.9135

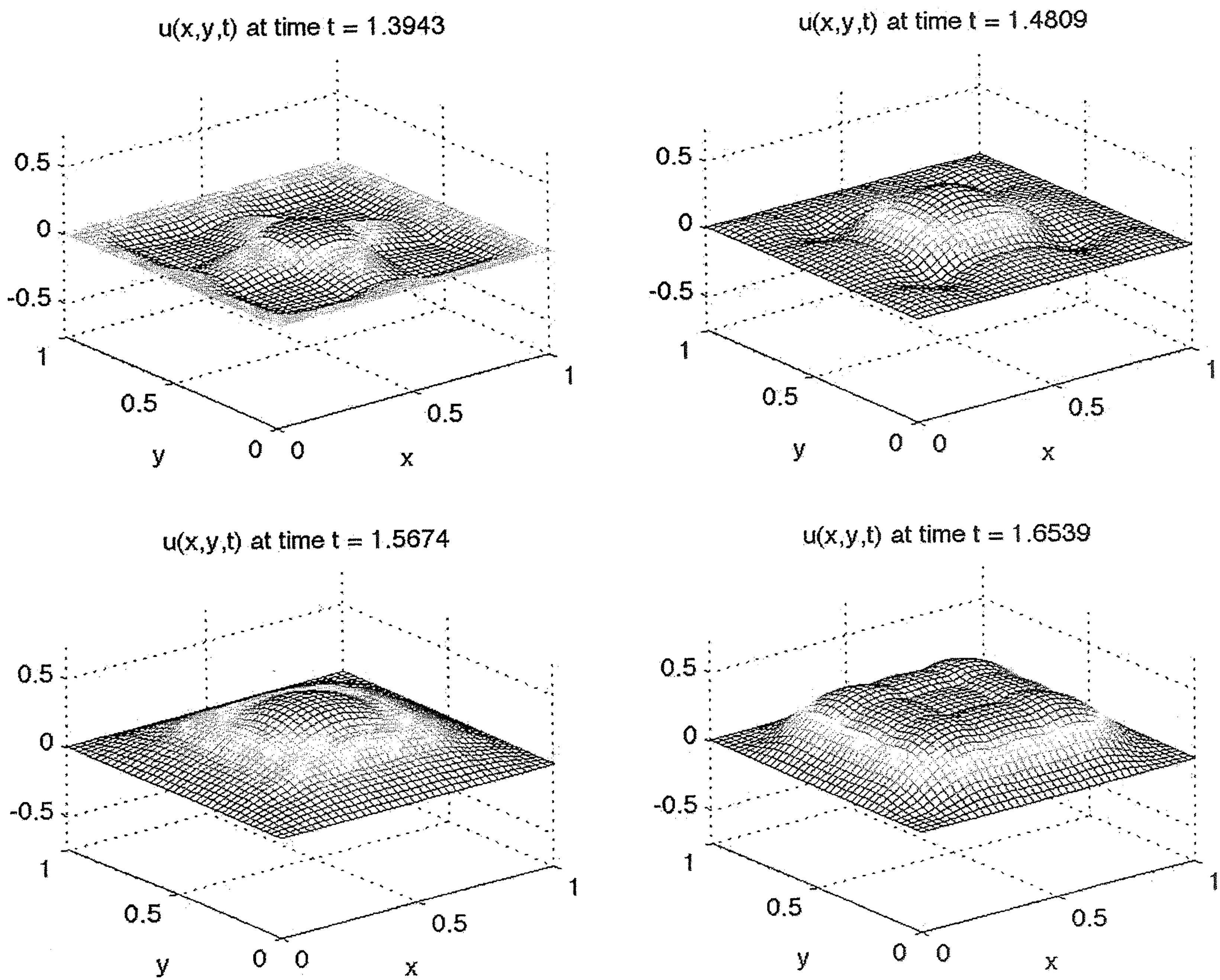


u(x,y,t) at time t = 2




u(x,y,t) at time t = 0.70217
u(x,y,t) at time t = 0.7887
u(x,y,t) at time t = 0.87522
u(x,y,t) at time t = 0.96174
0.5
0
-0.5
1
0.5
0
y
x


u(x,y,t) at time t = 1.0483
u(x,y,t) at time t = 1.1348
u(x,y,t) at time t = 1.2213
u(x,y,t) at time t = 1.3078
0.5
0
-0.5
1
0.5
0
y
0
0.5
1
x


u(x,y,t) at time t = 1.3943
u(x,y,t) at time t = 1.4809
u(x,y,t) at time t = 1.5674
u(x,y,t) at time t = 1.6539
0.5
0
-0.5
1
0.5
0
y
x

```
clear;

mp = 50;  % number of terms for each infinite series

x = linspace(0,1,40);     % x-points in plot in x = 0..1
y = linspace(0,1,40);     % y-points in plot in x = 0..1
t = linspace(1/12,2,24); % time points

% First compute e-values and coefficients
for n=1:mp
  for m=1:mp
    lambda(n,m) = (n*pi)^2 + (m*pi)^2;
    B(n,m) = (4/(pi*sqrt(lambda(n,m))))*((1/n)*(cos(3*pi*n/4) - cos(pi*n/4))..
         *(1/m)*(cos(3*pi*m/4) - cos(pi*m/4)));
  end
end

% Compute full solution at all points in space and time
for k=1:length(t)
  for i=1:length(x)
    for j=1:length(y)

       u(i,j,k) = 0;

       for n=1:mp
     for m=1:mp
       u(i,j,k) = u(i,j,k) + B(n,m)*sin(n*pi*x(i))*sin(m*pi*y(j))*..
            sin(sqrt(lambda(n,m))*t(k));
     end
       end

    end
  end
end

% Create the plots
mcount = 0;
for k=1:6
  figure(k)
  clf

  for m=1:4
    mcount = mcount + 1;
    subplot(2,2,m)
    mesh(x,y,u(:,:,mcount));
    colormap('jet');
    set(gca,'fontsize',10);
    title([ 'u(x,y,t) at time t = ',num2str(t(mcount))]);
    xlabel('x');
    ylabel('y');
    set(gca,'fontsize',10);
    axis([0 1 0 1 -0.75 0.75]);
  end
end

% Print the plots to JPG files
figure(1); print -djpeg95 memb1.jpg
figure(2); print -djpeg95 memb2.jpg
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```
figure(3); print -djpeg95 memb3.jpg
figure(4); print -djpeg95 memb4.jpg
figure(5); print -djpeg95 memb5.jpg
figure(6); print -djpeg95 memb6.jpg
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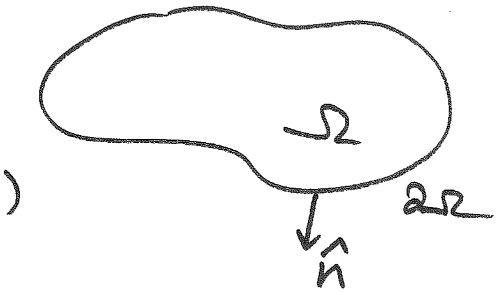
Multidimensional E'value Problems

After applying separation of variables to heat or wave equations, we obtained the following multidimensional e'value problem:



$$\nabla^2 \phi + \lambda \phi = 0 \qquad (x, y) \in \Omega \qquad (1)$$

$$\beta_1 \phi + \beta_2 \nabla\phi \cdot \hat{n} = 0 \qquad (x, y) \in \partial\Omega$$

$\hat{n}$: outward unit normal to $\partial\Omega$

Eq$\underline{^n}$ (1) is called Helmholtz equation. It is a partial case of the regular Sturm-Liouville problem:

$$\nabla(p\nabla\phi) + q\phi + \lambda\sigma\phi = 0$$

Helmholtz eq^n: $p=1$, $q=0$, $\sigma=1$

Thm (Helmholtz eq^n)

1. all e'values are real.
2. There is an infinite sequence of e'values. There is the smallest e'value and no largest e'value.
3. Corresponding to an e'value, there may be more than one associated e'functions (recall, in 1D, for a regular Sturm-Liouville problem, every e'value had only one associated e'function).

4. E'functions $\phi_\lambda(x,y)$ form a "complete set", namely, any piecewise smooth function $f(x,y)$ can be written as a generalized Fourier series:

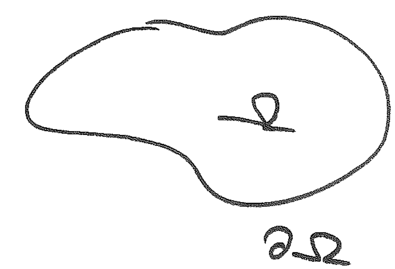
$$f(x,y) \sim \sum_\lambda a_\lambda \phi_\lambda(x,y) \qquad (2)$$

5. E'functions corresponding to distinct e'values are orthogonal:

$$\iint_\Omega \phi_{\lambda_1} \cdot \phi_{\lambda_2} \, dx\, dy = 0 \qquad \lambda_1 \neq \lambda_2$$

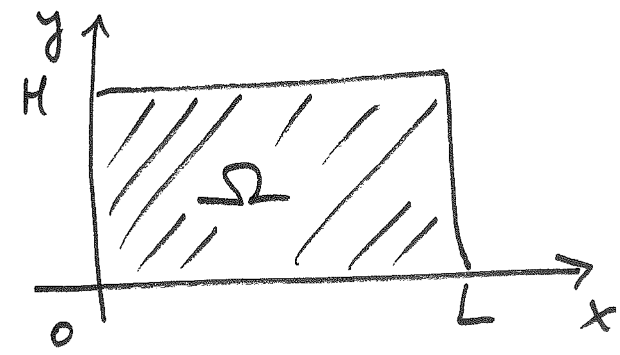
Furthermore, different linearly independent e'functions corresponding to the same e'value can be made orthogonal using Gram-Schmidt orthogonalization method. Hence, result (2) holds for e'functions with $\lambda_1 \neq \lambda_2$.

6. An e'value is related to the associated e'function by Rayleigh quotient:



$$\lambda = \frac{-\oint_{\partial\Omega} \phi \nabla\phi \cdot \hat{n} \, ds + \iint_{\Omega} |\nabla\phi|^2 \, dx\, dy}{\iint_{\Omega} \phi^2 \, dx\, dy}$$

Ex Vibrating rectangular membrane.



$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \quad : \quad \text{real} \quad n, m = 1, 2, \ldots$$

$$\lambda_{min} = \lambda_{11} = \left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{H}\right)^2$$

3. Multiple e'values.

In general, it is possible to have more than one e'function associated w/ the same e'value.

Ex Let $L = 2H$

$$\lambda_{nm} = \left(\frac{n\pi}{2H}\right)^2 + \left(\frac{m\pi}{H}\right)^2 = \left(\frac{\pi}{H}\right)^2 \left[\frac{n^2}{4} + m^2\right] = \frac{\pi^2}{4H^2}\left(n^2 + 4m^2\right)$$

$$\Phi_{nm}(x,y) = \sin\frac{n\pi x}{2H}\sin\frac{m\pi y}{H}$$

We can see that we can get different pairs of (n,m) that would give the same value λ_{nm} but different e'functions Φ_{nm}.

$n=2,\ m=1$

$4 + 4 = 8$?

$n=k,\ m=\frac{k}{2}$?

$k^2 + 4\cdot\frac{k^2}{4} = 2k^2$

$n=4,\ m=1$

$4^2 + 4\cdot 1 = 20$ ✓

$n=2,\ m=2$

$2^2 + 4\cdot 2^2 = 20$ ✓

$$\left.\begin{array}{l} n=4 \\ m=1 \end{array}\right\} \quad \lambda_{41} = \frac{\pi^2}{4H^2}\cdot 20$$

$$\Phi_{41}(x,y) = \sin\frac{4\pi x}{2H}\sin\frac{\pi y}{H} = \sin\frac{2\pi x}{H}\sin\frac{\pi y}{H}$$

$$\left.\begin{array}{l} n=2 \\ m=2 \end{array}\right\} \quad \lambda_{22} = \frac{\pi^2}{4H}\cdot 20 = \lambda_{41}$$

$$\Phi_{22}(x,y) = \sin\frac{2\pi x}{2H}\sin\frac{2\pi y}{H} = \sin\frac{\pi x}{H}\sin\frac{2\pi y}{H} \neq \Phi_{41}(x,y)$$