Lecture 3

INITIAL AND BOUNDARY CONDITIONS

Since the heat eq^{n} is 1st order in time and 2nd order in space, we need 1 initial condition and 2 boundary conditions.

<u>Initial condition:</u> $u(x,0) = f(x)$: <u>prescribed initial temperature</u>

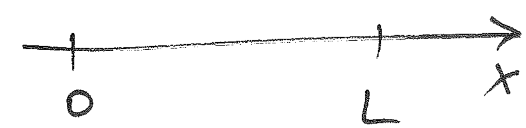
IC

<u>3 types of boundary conditions</u> (BCs)

1. <u>Dirichlet BC</u> : prescribed temperature

$u(0,t) = u_B(t)$

at $x=0$



Here $u_B(t)$ is a given temperature of the surrounding medium, e.g. we have a rod placed in some medium w/ ~~temperature~~ $u_B(t)$.

2. <u>Neumann BC</u>: prescribed gradient / flux

$$\underbrace{-K_0(0)\left.\frac{\partial u}{\partial x}\right|_{x=0}}_{\text{heat flux at } x=0} = \underbrace{\phi(t)}_{\text{flux}} : \text{ given function}$$

If $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \Rightarrow$ no heat flux through the boundary. $\Rightarrow$ we have "perfectly insulated" boundary at $x=0$

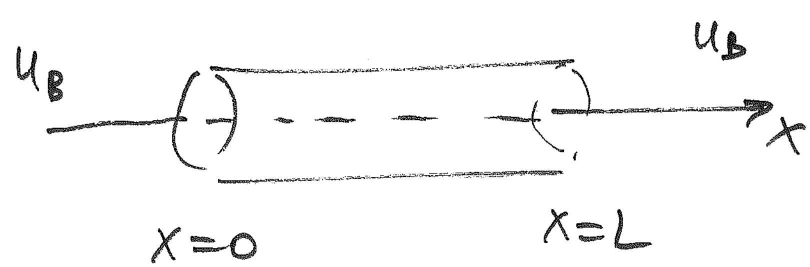
3. **Mixed BC**: **Newton's Law of Cooling**

at $x=0$: $\displaystyle -K_0(0)\frac{\partial u}{\partial x}\Big|_{x=0} = -H\Big[\underbrace{u(0,t) - u_B(t)}_{\text{difference of temperature at } x=0 \text{ and temperature } u_B \text{ of the surrounding medium}}\Big]$

heat flux at $x=0$

Here $H>0$: convection coefficient / heat transfer coefficient

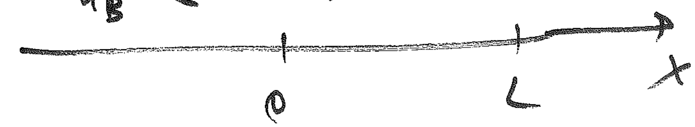
at $x=L$: $\displaystyle -K_0(L)\frac{\partial u}{\partial x}\Big|_{x=L} = H\big[u(L,t) - u_B(t)\big]$



The sign on the RHS is chosen so that if $u(0,t) > u_B(t)$, then the heat flows to the left, i.e. out the rod. Flux is positive if it is to the right, since here flux is negative. If $u(L,t) > u_B(t)$, the heat flows to the right, i.e. flux is positive.

<u>Another explanation</u> : let $u(0,t) > u_B(t)$

$u_B < u(0,t)$



$\Rightarrow$ we expect $u \nearrow$ as $x \nearrow$

$\Rightarrow \dfrac{\partial u}{\partial x} > 0$

<u>Meaning</u>: if $H \neq 0$, then this BC models NON-PEPFECTLY INSULATED boundary. In this case heat is lost / gained through the boundary until $u(x,t) \to u_B(t)$.

STEADY-STATE TEMPERATURE DISTRIBUTION

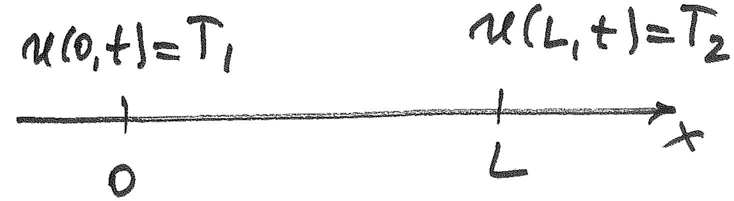
1. Prescribed temperature BC

Assume that all thermal coefficient / material properties c, ρ, K_0 are constant $\Rightarrow$ we can define $K = \frac{K_0}{c\rho}$: thermal diffusivity. Also assume that there are no sources/sinks: $Q = 0$

$$u_t = K u_{xx} \qquad 0 \leq x \leq L$$

$$u(x,0) = f(x) \qquad \text{IC}$$

$$\left.\begin{aligned} u(0,t) &= T_1 \\ u(L,t) &= T_2 \end{aligned}\right\} \text{BCs}$$



Def a <u>steady-state solution</u> is a time-independent solution. $\frac{\partial u}{\partial t} = 0$

Another way to define a steady-state solution is

$$\lim_{t \to \infty} u(x,t)$$

Let's consider the above problem w/ T_1 and T_2 being constants and find the steady-state solution

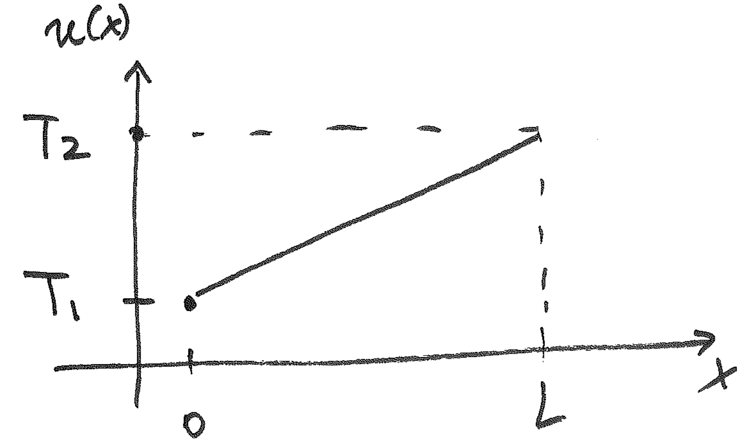
Steady-state: $\frac{\partial u}{\partial t} = 0 \Rightarrow u = u(x)$

$$\cancelto{0}{u_t} = \underbrace{K}_{\neq 0} u_{xx} \Rightarrow u_{xx} = 0 \Leftrightarrow \frac{d^2 u}{dx^2} = 0$$ this is a 2^{nd} order ODE

Solution: $u(x) = C_1 x + C_2$

We can apply BCs at $x=0$ and $x=L$ to find C_1, C_2:

$$u(x) = T_1 + \frac{T_2 - T_1}{L}\, x$$



Note: If we consider time-dependent problem, $u(x,t)$ will change in time. But if BCs are still time-independent, we expect that solution $u(x,t)$ would approach the steady-state solution as $t \to \infty$, i.e.

$$u(x,t) \to u(x) = T_1 + \frac{T_2 - T_1}{L}\, x \quad \text{as } t \to \infty$$

Note: we did not use IC here.

2. Insulated boundaries

$u_t = K u_{xx} \qquad 0 \le x \le L$

IC $\quad u(x,0) = f(x)$

BCs $\quad u_x(0,t) = u_x(L,t) = 0$

We need to find the steady-state solution.

$u_t = 0 \Rightarrow u_{xx} = 0 \Rightarrow u(x) = C_1 x + C_2$

$u'(x) = C_1$

$u'(0) = u'(L) = 0$

$u'(0) = 0 \Rightarrow C_1 = 0 \Rightarrow \boxed{u(x) = C_2}$: steady-state temperature defined up to an arbitrary constant C_2

To find C_2, we will use the thermal energy conservation law in the integral form.

let $a=0$, $b=L$

$$\frac{d}{dt}\int_0^L c\rho\, u(x,t)\,dx = \overset{0}{\phi(0,t)} - \overset{0}{\phi(L,t)} = 0$$

$$\phi(0,t) = \underbrace{-K_0(0)\frac{\partial u}{\partial x}\Big|_{x=0}}_{=0} \qquad \phi(L,t) = \underbrace{-K_0(L)\frac{\partial u}{\partial x}\Big|_{x=L}}_{=0}$$

$$\frac{d}{dt}\int_0^L c\rho\, u(x,t)\,dx = 0 \Rightarrow \int_0^L c\rho\, u(x,t)\,dx = \text{const}$$

$$\Rightarrow \int_0^L c\rho\, \underbrace{u(x,0)}_{f(x)}\,dx = \int_0^L c\rho\, u(x,t)\,dx = \int_0^L c\rho\, \overbrace{u(x)}^{=C_2}\,dx$$

any time

steady-state solution

$c = \text{const}, \quad \rho = \text{const}$

$\int_0^L \, dx$

$$\int_0^L f(x)\,dx = C_2 \cdot L$$

$$\Rightarrow \boxed{C_2 = \frac{1}{L}\int_0^L f(x)\,dx} :$$ average initial temperature

Hence,

$$\lim_{t\to\infty} u(x,t) = u(x) = C_2 = \frac{1}{L}\int_0^L f(x)\,dx :$$ steady-state solution for insulated boundaries