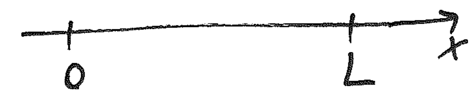
Lecture 3

3 types of boundary conditions

1. **Dirichlet BC** : prescribed temperature



$u(x,t)$

$u(0,t) = u_B(t)$ where $u_B(t)$ is a given temperature of surrounding medium, eg.

at $x=0$

we have a rod placed in some medium of temperature $u_B(t)$

2. **Neumann BC** : prescribed gradient / flux

$$\underbrace{-K_0(0)\frac{\partial u}{\partial x}\Big|_{x=0}}_{\text{heat flux at } x=0} = \underbrace{\phi(t)}_{\text{flux}}$$ where $\phi(t)$ is given

If $\frac{\partial u}{\partial x}\Big|_{x=0} = 0 \Rightarrow$ no heat flux through boundary

$\Rightarrow$ "perfectly insulated" boundary at $x=0$

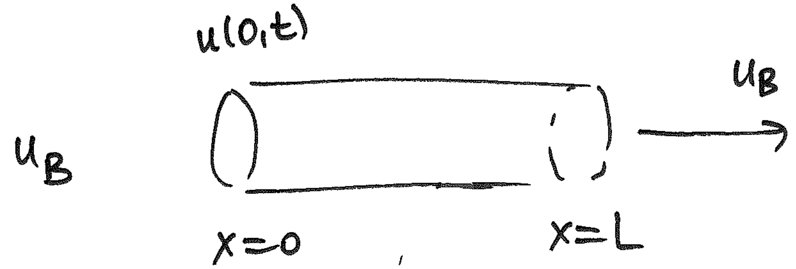
3. Mixed BC : Newton's Law of Cooling

at $x=0$: $\quad \underbrace{-K_0(0)\dfrac{\partial u}{\partial x}\Big|_{x=0}}_{\text{heat flux}} = -H\big[\underbrace{u(0,t) - u_B(t)}_{\substack{\text{difference of temperature} \\ \text{of the rod at } x=0 \text{ and} \\ \text{temp. } u_B \text{ of surrounding medium}}}\big]$

Here $H>0$: convection coefficient / heat transfer coefficient

$u_B(t)$: temperature of surrounding medium

at $x=L$: $\quad -K_0(L)\dfrac{\partial u}{\partial x}\Big|_{x=L} = H\big[u(L,t) - u_B(t)\big]$



The sign on RHS is chosen so that if $u(0,t) > u_B(t)$, then the heat flows out the rod, i.e. to the left, hence, the flux at $x=0$ is negative. If $u(L,t) > u_B(t)$, the heat flows to the right, i.e. the heat flux is positive.

Another explanation: $u(0,t) > u_B$

$u(0,t)$

u_B to the left from $x=0$, at some x to the left of $x=0$ $x=0$

$\Rightarrow$ we expect $u \nearrow$ as $x \nearrow$

$\Rightarrow \left.\dfrac{\partial u}{\partial x}\right|_{x=0} > 0$

Meaning: if $H \neq 0$, then this BC models NON-PERFECTLY INSULATED boundary $\rightarrow$ heat is lost/gained through the boundary until $u(x,t) \rightarrow u_B(t)$.

STEADY-STATE TEMPERATURE DISTRIBUTION

1. Prescribed temperature BC

Assume that all thermal coefficients c, ρ, K_0 are constant

$\Rightarrow$ we can define $K = \dfrac{K_0}{c\rho}$: thermal diffusivity coefficient

Also assume that there are no sources or sinks $\rightarrow Q = 0$

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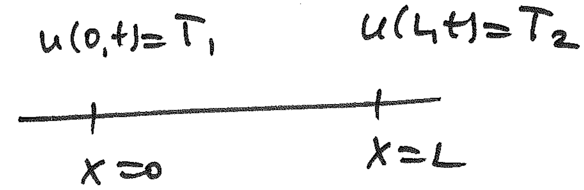
$u_t = K u_{xx} \qquad 0 \le x \le L$



$u(x,0) = f(x)$ IC

$$\left.\begin{aligned} u(0,t) &= T_1 \\ u(L,t) &= T_2 \end{aligned}\right\} \text{BCs}$$

Def a steady-state solution is a solution that does not depend on time.

Another way to define a steady-state solution is

$$\lim_{t\to\infty} u(x,t)$$

Let's solve the above problem for T_1, T_2 being constants and find the steady-state solution.

Steady-state $\Rightarrow u_t = 0 \Rightarrow u_{xx} = 0$

$u(x,\cancel{t}) = u(x)$

$\Rightarrow u_{xx} = 0 \iff u'' = 0 \,, \quad u = u(x)$

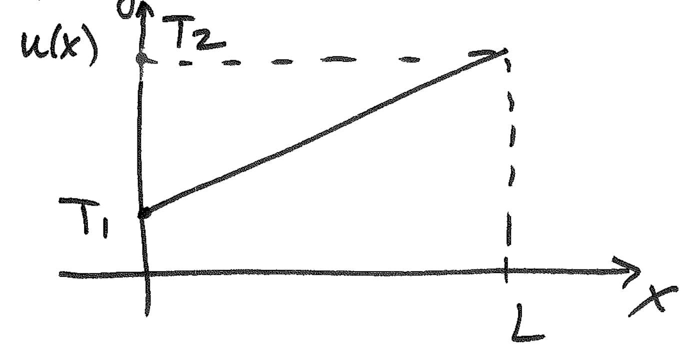
$' = \frac{d}{dx}$

This is a 2nd order ODE

$u(x) = C_1 x + C_2$

Using BCs at $x=0$ and $x=L$, we get

$$u(x) = T_1 + \frac{T_2 - T_1}{L} x$$



Now, if we consider time-dependent problem, $u(x,t)$ will change in time. But if BCs are still independent of time, we expect that solution $u(x,t)$ approaches the steady-state solution as $t \to \infty$

$$u(x) = T_1 + \frac{T_2 - T_1}{L} x$$

Note We have not used IC here.

2. Insulated boundaries

$u_t = k u_{xx} \qquad 0 \leq x \leq L$

IC $\quad u(x,0) = f(x)$

BCs $\quad u_x(0,t) = u_x(L,t) = 0$

We need to find the steady-state solution.

$u_t = 0 \Rightarrow u_{xx} = 0 \Rightarrow u(x) = C_1 x + C_2$

$u'(0) = u'(L) = 0 \qquad$ BCs

$u'(x) = C_1 \Rightarrow u'(0) = 0 \Rightarrow C_1 = 0$

$\Rightarrow u(x) = C_2$: steady-state solution where C_2 is an arbitrary constant

To find C_2, we will use the thermal energy conservation law in the integral form

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Let $a=0, \quad b=L.$

$$\frac{d}{dt}\int_0^L c\rho u(x,t)\,dx = \overset{0}{\underset{-K_0(0)\underbrace{\frac{\partial u}{\partial x}(0,t)}_{=0}}{\phi(0,t)}} - \overset{0}{\underset{-K_0(L)\underbrace{\frac{\partial u}{\partial x}(L,t)}_{=0}}{\phi(L,t)}} = 0$$

$$\frac{d}{dt}\int_0^L c\rho u(x,t)\,dx = 0 \Rightarrow \int_0^L c\rho u(x,t)\,dx = \text{const}$$

$$\Rightarrow \int_0^L c\rho \underbrace{\overbrace{u(x,0)}^{\text{temp. at } t=0}}_{f(x)}\,dx = \int_0^L c\rho \underbrace{\overset{\text{steady-state}}{\underset{t\to\infty}{u(x)}}}_{=C_2}\,dx = \int_0^L c\rho \underset{\text{any } t}{u(x,t)}\,dx$$

$c=$ const

$\rho=$ const

$$\Rightarrow \int_0^L f(x)\,dx = C_2 \cdot L$$

$$\Rightarrow \boxed{C_2 = \frac{1}{L}\int_0^L f(x)\,dx} :$$

average initial temperature

Hence,

$$\lim_{t\to\infty} u(x,t) = u(x) = c_2 = \frac{1}{L}\int_0^L f(x)\,dx :$$

steady-state solution for insulated boundaries