#2
HW #11

$$f(x) = \begin{cases} x & \text{if } x \le \frac{1}{2} \\ 1-x & \text{if } x \ge \frac{1}{2} \end{cases} \qquad 0 \le x \le 1$$

$$\text{Rel Error }(M) = \frac{\left| \int_a^b f^2 \sigma\, dx - \sum_{n=1}^{M} a_n^2 \int_a^b \phi_n^2 \sigma\, dx \right|}{\left| \int_a^b f^2 \sigma\, dx \right|} < 10^{-4} \quad \text{etc.}$$

Here $a=0$, $b=1$.

Fourier sine series: $\sigma = 1$, $\phi_n(x) = \sin \frac{n\pi x}{1}$

let $\int_0^1 f^2\, dx \equiv A$

$$\text{Rel Error }(M) = \frac{\left| A - \sum_{n=1}^{M} a_n^2 \cdot \frac{1}{2} \right|}{|A|} < \text{tolerance}$$

let $M=5 \Rightarrow$ Rel. Err $= \ldots$ find M that would work w/ 10^{-4}, etc.

Rel Err $= 10^{-4} = E_1 \Rightarrow M = M_1$ find M_1

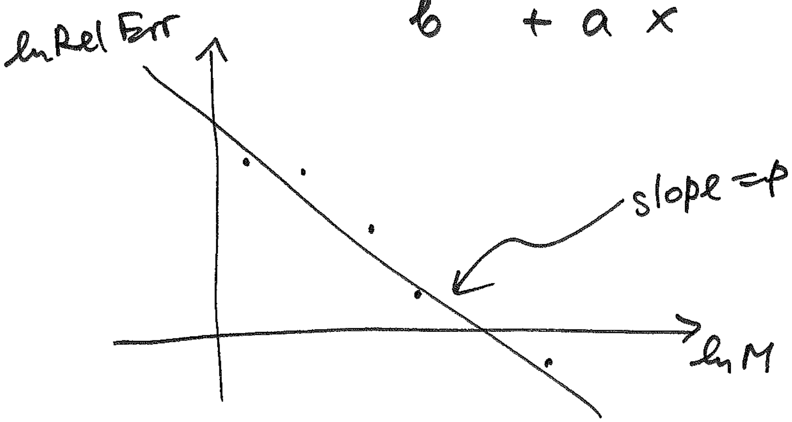
Rel Err $= 10^{-5} = E_2 \Rightarrow M = M_2$ find M_2

$= 10^{-6}$ $M = M_3$ - - -

10^{-7} $M = M_4$ - - -

$\text{Rel Err} = C M^p$ $\quad C-?\quad p-?$

$\underbrace{\ln \text{Rel Err}} = \underset{b}{\ln C} + \underset{a}{p}\,\underbrace{\ln M}_{x}$



assume $C \approx 1 \Rightarrow \ln C \approx 0$

$\Rightarrow \ln \text{Rel Err} \approx p \ln M$

$\Rightarrow p \approx \dfrac{\ln \text{Rel Err}}{\ln M}$

Evaluate $\dfrac{\ln E_1}{\ln M_1}, \dfrac{\ln E_2}{\ln M_2}, \dfrac{\ln E_3}{\ln M_3}, \dfrac{\ln E_4}{\ln M_4} \sim p$

Alternatively, use least-squares fitting to find p

$\underset{y}{\ln \text{Rel Err}} \sim \underset{a}{p}\,\underbrace{\ln M}_{x} + \underbrace{\ln C}_{b}$ $\qquad y \sim ax + b$

Thm (Helmholtz equation)

1. All e'values are real.
2. There is an infinite sequence of e'values. There is the smallest e'value and no largest e'value.

3. Corresponding to an e'value, there may be more than one e'function (recall, in 1D, for a regular Sturm-Liouville problem, every e'value had only one e'value.)

4. E'functions $\phi_\lambda(x,y)$ form a "complete set", namely, any piecewise smooth function can be written as a generalized Fourier series:

$$f(x,y) \sim \sum_\lambda a_\lambda \phi_\lambda(x) \qquad (*)$$

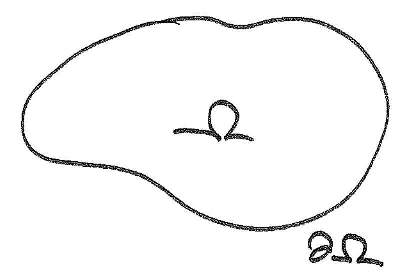
5. E'functions corresponding to distinct e'values are orthogonal:

$$\iint_{\Omega} \phi_{\lambda_1} \phi_{\lambda_2} \, dx\, dy = 0 \qquad \lambda_1 \neq \lambda_2$$

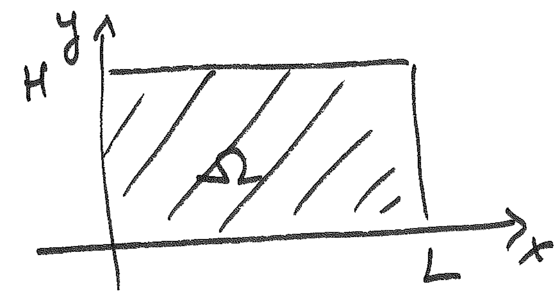
Furthermore, different linearly independent e'functions corresponding to the same e'value can be made orthogonal using Gram – Schidt orthogonalization method. Hence (*) holds for e'functions with $\lambda_1 = \lambda_2$.

6. An e'value is related to the associated e'function by Rayleigh Quotient:

$$\lambda = \frac{-\oint_{\partial\Omega} \phi \nabla\phi \cdot \hat{n}\, dS + \iint_{\Omega} |\nabla\phi|^2 \, dx\, dy}{\iint_{\Omega} \phi^2 \, dx\, dy}$$



Ex Vibrating rectangular membrane.



$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \; : \text{ real}$$

$n, m = 1, 2, \ldots$

$$\lambda_{min} = \lambda_{11} = \left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{H}\right)^2$$

3. <u>Multiple eigenvalues</u>

In general, it is possible to have more than one e'function associated w/ the same e'value.

Ex Let $L = 2H$

$$\lambda_{nm} = \left(\frac{n\pi}{2H}\right)^2 + \left(\frac{m\pi}{H}\right)^2 = \frac{\pi^2}{4H^2}\left(n^2 + 4m^2\right)$$

$$\phi_{nm}(x,y) = \sin\frac{n\pi x}{2H}\,\sin\frac{m\pi y}{H}$$

we can see that we can get different pairs of (n, m) that would give the same λ_{nm} but different e'functions ϕ_{nm}.

$$\left.\begin{array}{l} n=4 \\ m=1 \end{array}\right\} \Rightarrow \lambda_{41} = \frac{\pi^2}{4H^2} \cdot 20$$

$$\phi_{41}(x,y) = \sin\frac{4\pi x}{2H} \sin\frac{\pi y}{H} = \sin\frac{2\pi x}{H} \sin\frac{\pi y}{H}$$

$$\left.\begin{array}{l} n=2 \\ m=2 \end{array}\right\} \Rightarrow \lambda_{22} = \frac{\pi^2}{4H^2} \cdot 20$$

$$\phi_{22}(x,y) = \sin\frac{2\pi x}{2H} \sin\frac{2\pi y}{H} = \sin\frac{\pi x}{H} \sin\frac{2\pi y}{H}$$

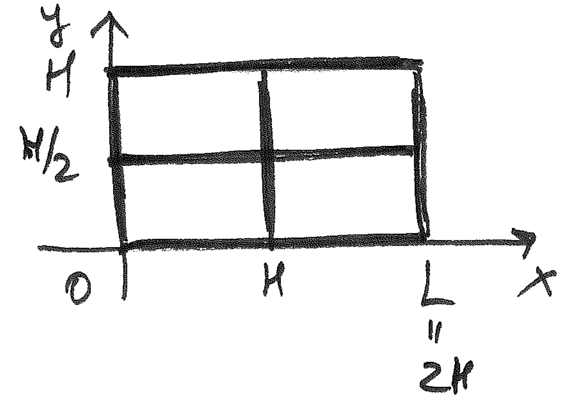
<u>Nodal curves</u>: $\phi = 0$

<u>Ex</u> $n=2$, $m=2$ $\qquad \phi_{22}(x,y) = \sin\frac{\pi x}{H} \sin\frac{2\pi y}{H}$

$$\phi_{22} = 0 \Rightarrow \sin\frac{\pi x}{H} = 0 \quad \text{or} \quad \sin\frac{2\pi y}{H} = 0$$

$\sin\frac{\pi x}{H} = 0 \rightarrow \frac{\pi x}{H} = k\pi \Rightarrow x = kH$

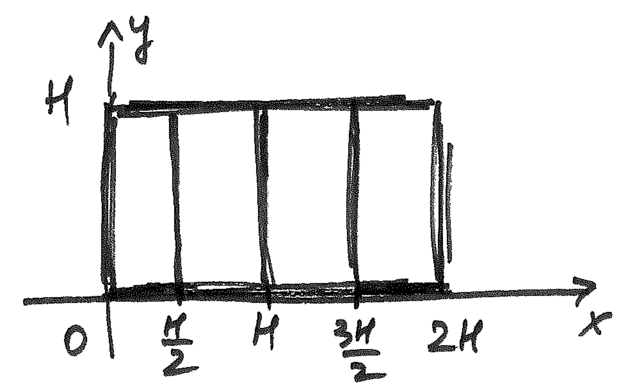
$\Rightarrow x = 0,\ H,\ 2H \quad (k = 0, 1, 2)$



$\sin\frac{2\pi y}{H} = 0 \Rightarrow \frac{2\pi y}{H} = l\pi \Rightarrow y = \frac{lH}{2}$

$\Rightarrow y = 0,\ \frac{H}{2},\ H \quad (l = 0, 1, 2)$

Ex $n = 4,\ m = 1$ $\quad \phi_{41}(x,y) = \sin\frac{2\pi x}{H}\sin\frac{\pi y}{H}$



$\sin\frac{2\pi x}{H} = 0 \Rightarrow \frac{2\pi x}{H} = k\pi \Rightarrow x = \frac{kH}{2}$

$x = 0,\ \frac{H}{2},\ H,\ \frac{3H}{2},\ 2H$

$\sin\frac{\pi y}{H} = 0 \Rightarrow \frac{\pi y}{H} = l\pi \Rightarrow y = lH$

$\Rightarrow y = 0,\ H$

Note Sets of nodal curve of ϕ_{22} and ϕ_{41} are different $\Rightarrow$ e'functions ϕ_{22} and ϕ_{41} are different

<u>Note</u> we can write $\lambda_{22} = \lambda_{41}$

In general, $\lambda_{(2n)m} = \lambda_{(2m)n}$ (by symmetry)

but $\phi_{2n,m} \neq \phi_{2m,n}$

We can get the same e'value w/o symmetry :

<u>Ex</u> $n=7,\ m=2 \Rightarrow \lambda_{72} = \dfrac{\pi^2}{4H^2}(7^2 + 4\cdot 2^2) = \dfrac{\pi^2}{4H^2}\,65$

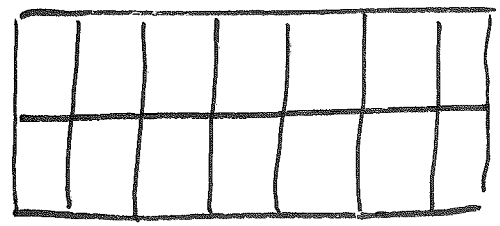
$$\phi_{72}(x,y) = \sin\frac{7\pi x}{2H}\sin\frac{2\pi y}{H}$$

$n=1,\ m=4 \Rightarrow \lambda_{14} = \dfrac{\pi^2}{4H^2}\,65$

$$\phi_{14}(x,y) = \sin\frac{\pi x}{2H}\sin\frac{4\pi y}{H}$$

Nodal curves

$n=1, \; m=4$



$n=7, \; m=2$

Ex Another example : $\lambda_{28} = \lambda_{(16)1} = \lambda_{(14)4} = \lambda_{87} = \frac{\pi^2}{4H} 260$

$\therefore$ 4 different e'functions $\phi_{2,8}$, $\phi_{16,1}$, $\phi_{14,4}$, ϕ_{87} correspond to the same e'value

4 Series of e'functions

Any piecewise smooth function $f(x,y)$ can be written as

$$f(x,y) \sim \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \phi_{nm}(x,y)$$

In general

$$f(x,y) \sim \sum_{\lambda} a_\lambda \phi_\lambda(x)$$

<u>Convergence</u>: to approximate $f(x,y)$ we can use a truncated generalized Fourier series. To estimate an error, we can introduce mean-square deviation:

$$E = \iint_{\Omega} \left(f - \sum_{\lambda}^{M} a_\lambda \phi_\lambda\right)^2 dx\,dy$$

E is minimized when a_λ are Fourier coefficients.

$E \to 0$ as we use all e'values and e'functions.