Lecture 41

5 Orthogonality of eigenfunctions

$$f(x,y) \sim \sum_{\lambda} a_\lambda \phi_\lambda(x,y) \quad \Big/ \cdot \phi_{\lambda_i} \qquad \iint_\Omega$$

$$\Rightarrow \iint_\Omega f \phi_{\lambda_i} \, dx\,dy = \sum_\lambda a_\lambda \iint_\Omega \phi_\lambda \phi_{\lambda_i} \, dx\,dy$$

by orthogonality, $\iint_\Omega \phi_\lambda \phi_{\lambda_i} \, dx\,dy = 0$ if $\lambda \neq \lambda_i$

$$\therefore \quad \boxed{a_{\lambda_i} = \frac{\iint_\Omega f \phi_{\lambda_i} \, dx\,dy}{\iint_\Omega \phi_{\lambda_i}^2 \, dx\,dy}}$$

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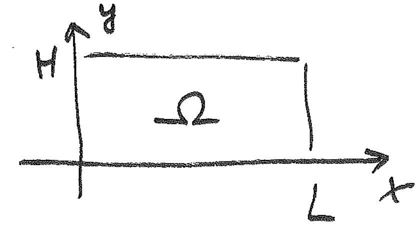
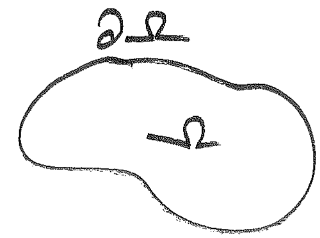
<u>Special case</u> (rectangular membrane)



$$a_{nm} = \frac{\int_0^H \int_0^L f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dx\, dy}{\int_0^H \int_0^L \sin^2\frac{n\pi x}{L} \sin^2\frac{m\pi y}{H}\, dx\, dy}$$

$$\therefore \quad \boxed{a_{nm} = \frac{4}{HL} \int_0^H \int_0^L f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dx\, dy}$$

Multidimensional Eigenvalue Problems



Recall Helmholtz equation:

$\nabla^2 \phi + \lambda \phi = 0$ w/ $\beta_1 \phi + \beta_2 \nabla \phi \cdot \hat{n} = 0$ on $\partial\Omega$

Let $\mathcal{L} = \nabla^2$: differential operator $\Rightarrow$ $\mathcal{L}\phi + \lambda\phi = 0$

multidimensional Green's formula

Let u and v be two functions of x and y.

$$\Rightarrow u\,\mathcal{L}(v) - v\,\mathcal{L}(u) = u\nabla^2 v - v\nabla^2 u$$

Note:
$$\nabla\cdot(u\nabla v) = \nabla u\cdot\nabla v + u\nabla^2 v \quad -$$
$$\nabla\cdot(v\nabla u) = \nabla v\cdot\nabla u + v\nabla^2 u$$

$$\therefore \boxed{u\nabla^2 v - v\nabla^2 u = \nabla\cdot(u\nabla v - v\nabla u)}$$ multi-dimensional Lagrange identity

$$\iint_\Omega [u\nabla^2 v - v\nabla^2 u]\,dx\,dy = \iint_\Omega \nabla\cdot(u\nabla v - v\nabla u)\,dx\,dy$$

Note: divergence thm: $\displaystyle\iint_\Omega \nabla\cdot\vec{A}\,dx\,dy = \int_{\partial\Omega} \vec{A}\cdot\vec{n}\,dS$

$\therefore$ Green's formula:

$$\boxed{\iint_\Omega [u\nabla^2 v - v\nabla^2 u]\,dx\,dy = \oint_{\partial\Omega} (u\nabla v - v\nabla u)\cdot\hat{n}\,dS}$$

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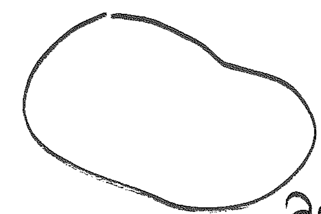
Def The operator $\mathcal{L} = \nabla^2$ is <u>self-adjoint</u> in the following sense:

If u and v are two functions such that

$$\oint_{\partial\Omega} (u \nabla v - v \nabla u)\, \vec{n}\, ds = 0 \qquad (*)$$



then

$$\iint_{\Omega} [\underbrace{u \nabla^2 v - v \nabla^2 u}_{u\mathcal{L}(v) - v\mathcal{L}(u)}]\, dx\, dy = 0$$

Note: $(*)$ is satisfied by BC $\beta_1 \phi + \beta_2 \nabla \phi \cdot \hat{n} = 0$ on $\partial\Omega$ ✓

1. $\beta_2 = 0 \Rightarrow \phi = 0$ on $\partial\Omega$ $\Rightarrow u = 0, v = 0$ on $\partial\Omega$
2. $\beta_1 = 0 \Rightarrow \nabla u \cdot \hat{n} = 0,\ \nabla v \cdot \hat{n} = 0$ on $\partial\Omega$ ✓
3. $\beta_1 u + \beta_2 \nabla u \cdot \hat{n} = 0 \qquad \beta_1 v + \beta_2 \nabla v \cdot \hat{n} = 0$

$$\oint_{\partial\Omega} (u\nabla v - v\nabla u)\cdot\vec{n}\, ds \overset{?}{=} 0$$

$$\oint_{\partial\Omega} \Big(u \underbrace{\nabla v\cdot\vec{n}}_{= -\frac{\beta_1}{\beta_2} v} - v \underbrace{\nabla u\cdot\vec{n}}_{= -\frac{\beta_1}{\beta_2} u}\Big)\, ds = \oint_{\partial\Omega} \Big[\cancel{uv\big(-\tfrac{\beta_1}{\beta_2}\big)} + \cancel{\tfrac{\beta_1}{\beta_2} vu}\Big]\, ds = 0 \checkmark$$

<u>Claim</u>: Any two eigenfunctions corresponding to distinct e'values (λ_a, λ_b) are orthogonal.

<u>Proof</u>

$$\nabla^2\phi_a + \lambda_a\phi_a = 0 \quad \text{and} \quad \nabla^2\phi_b + \lambda_b\phi_b = 0$$

$$0 = \iint_{\Omega} \Big[\phi_a \underbrace{\nabla^2\phi_b}_{= -\lambda_b\phi_b} - \phi_b \underbrace{\nabla^2\phi_a}_{= -\lambda_a\phi_a}\Big]\, dx\,dy = \iint_{\Omega} (\lambda_a - \lambda_b)\,\phi_a\phi_b\, dx\,dy =$$

∇^2 is self-adjoint

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$$= \underbrace{(\lambda_a - \lambda_b)}_{\neq 0} \iint_\Omega \phi_a \phi_b \, dx\, dy \quad \Rightarrow \quad \iint_\Omega \phi_a \phi_b \, dx\, dy = 0$$

since λ_a, λ_b are distinct

$\therefore$ ϕ_a, ϕ_b are orthogonal w/ weight 1.

Claim eigenvalues of the Helmholtz equation w/ BC $\beta_1 \phi + \beta_2 \nabla\phi \cdot \hat{n} = 0$ on $\partial\Omega$ are real.

Proof (by contradiction)

Assume λ is complex w/ complex conjugate $\bar{\lambda}$.

Assume ϕ is complex w/ — " — $\bar{\phi}$

(λ, ϕ)

$\nabla^2 \phi + \lambda \phi = 0 \Rightarrow \nabla^2 \bar{\phi} + \bar{\lambda} \bar{\phi} = 0 \Rightarrow \bar{\lambda}$ is an e'value of Helmholtz eq$^{\underline{n}}$ w/ associated e'function $\bar{\phi}$.

By the same arguments as above:

$$(\lambda-\bar{\lambda})\iint_\Omega \phi\bar{\phi}\,dx\,dy = 0$$

$$\phi = \phi_r + i\,\phi_i \Rightarrow \phi\bar{\phi} = (\phi_r + i\,\phi_i)(\phi_r - i\,\phi_i) = \phi_r^2 + \phi_i^2 > 0$$

ϕ_r, ϕ_i real-valued functions

since $\phi \not\equiv 0$ as an e'function

$$\therefore \underbrace{\iint_\Omega \phi\bar{\phi}\,dx\,dy}_{\text{i.e. } \neq 0} > 0 \Rightarrow \lambda - \bar{\lambda} = 0 \Rightarrow \lambda \text{ is real}$$

Gram-Schmidt process

In multidimensional case, it is possible to have several e'functions corresponding to the same e'value. Suppose $\phi_1, \phi_2, \ldots, \phi_n$ correspond to λ. These functions are linearly independent but not necessarily mutually orthogonal.

Q: Can we make them orthogonal?

Key: any linear combination of $\phi_1, \phi_2, \dots, \phi_n$ is still an e'function corresponding to λ

Proof

Let $\psi = \sum_{i=1}^{n} c_i \phi_i$ where $\nabla^2 \phi_i + \lambda \phi_i = 0$

Because ∇^2 is a linear operator $\nabla^2 \left(\sum_{i=1}^{n} c_i \phi_i \right) + \lambda \sum_{i=1}^{n} c_i \phi_i = 0$

$\therefore$ ψ is an e'function corresponding to λ.

Gram - Schmidt

$\phi_1, \phi_2, \dots, \phi_n \rightarrow \psi_1, \psi_2 \dots \psi_n$

1. Let $\psi_1 = \phi_1$
2. Let $\psi_2 = \phi_2 + c_{11} \psi_1$ $\quad / \psi_1 \quad \iint_\Omega$

Want: $\iint_\Omega \psi_1 \psi_2 \, dx\,dy = 0$

$$\rightarrow \iint_\Omega \phi_2 \psi_1 \, dx\,dy + c_{11} \iint_\Omega \psi_1^2 \, dx\,dy = 0$$

$$\therefore \quad C_{11} = -\frac{\iint_\Omega \phi_2 \psi_1 \, dx\,dy}{\iint_\Omega \psi_1^2 \, dx\,dy}$$

3. Let $\psi_3 = \phi_3 + C_{21}\psi_1 + C_{22}\psi_2 \qquad / \psi_1 \qquad / \cdot \psi_2$

We want

$$\iint_\Omega \psi_1 \psi_3 \, dx\,dy = 0$$

$$\iint_\Omega \psi_2 \psi_3 \, dx\,dy = 0$$

$$\iint_\Omega \phi_3 \psi_1 \, dx\,dy + C_{21} \underbrace{\iint_\Omega \psi_1^2 \, dx\,dy} + C_{22} \iint_\Omega \cancelto{0}{\psi_1 \psi_2} \, dx\,dy = 0$$

$$C_{21} = -\frac{\iint_\Omega \phi_3 \psi_1 \, dx\,dy}{\iint_\Omega \psi_1^2 \, dx\,dy}$$

Similarly

$$c_{22} = -\frac{\iint_\Omega \phi_3 \psi_2 \, dx\, dy}{\iint_\Omega \psi_2^2 \, dx\, dy}$$

In general,

$$\boxed{\psi_j = \phi_j - \sum_{i=1}^{j-1} \underbrace{\frac{\iint_\Omega \phi_j \psi_i \, dx\, dy}{\iint_\Omega \psi_i^2 \, dx\, dy}}_{c_{ij}} \cdot \psi_i}$$

$\therefore$ $\psi_1, \psi_2, \ldots, \psi_n$ are mutually orthogonal and correspond to the same e'value λ