University of Idaho

Lecture 41

#2
HW #11

$$f(x)=\begin{cases} x & \text{if } x\le \frac{1}{2} \\ 1-x & \text{if } x\ge \frac{1}{2}\end{cases} \qquad 0\le x\le 1$$

$$\text{Rel Error}(M)=\frac{\left|\int_a^b f^2\sigma\,dx-\sum_{n=1}^{M}a_n^2\int_a^b \phi_n^2\sigma\,dx\right|}{\left|\int_a^b f^2\sigma\,dx\right|} < 10^{-4} \text{ etc.}$$

$$\int_0^1 \ldots dx=\int_0^{1/2}\ldots+\int_{1/2}^{1}\ldots$$

Here $a=0$, $b=1$.

Fourier sine series: $\sigma=1$, $\phi_n(x)=\sin\frac{n\pi x}{1}$

let $\int_0^1 f^2\,dx \equiv A$

$$\text{Rel Error}(M)=\frac{\left|A-\sum_{n=1}^{M}a_n^2\cdot\frac{1}{2}\right|}{|A|} < \text{tolerance}$$

find the smallest M that would work w/ 10^{-4}, etc.

let $M=5 \Rightarrow$ Rel. Err $=\ldots$

Rel Err $=10^{-4}=E_1 \Rightarrow M=M_1$ find M_1

Rel Err $=10^{-5}=E_2 \Rightarrow M=M_2$ find M_2

$=10^{-6}$ $M=M_3$ - - -

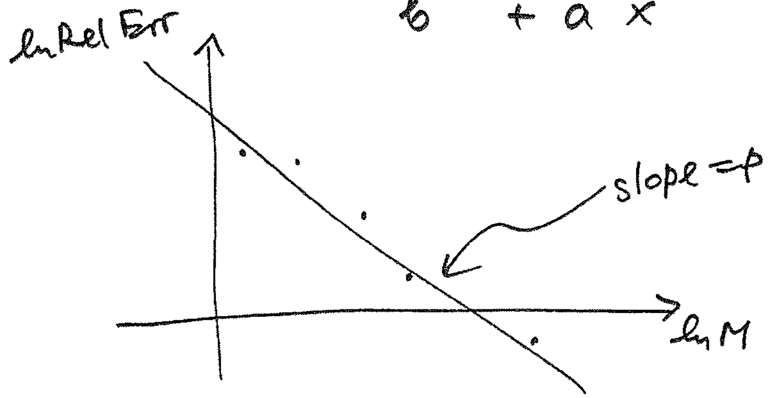
10^{-7} $M=M_4$ - - -

$\text{Rel Err} = C M^p \qquad C-? \qquad p-?$

$\ln \text{Rel Err}$ is a linear function of $\ln M$ w/ slope $= p$

$$\underbrace{\ln \text{Rel Err}}_{y} = \underbrace{\ln C}_{b} + p\,\underbrace{\ln M}_{x}$$

$$= b + a\,x$$



assume $C \simeq 1 \Rightarrow \ln C \approx 0$

$\Rightarrow \ln \text{Rel Err} \simeq p \ln M$

$\Rightarrow p \simeq \dfrac{\ln \text{Rel Err}}{\ln M}$

Evaluate $\dfrac{\ln E_1}{\ln M_1}, \dfrac{\ln E_2}{\ln M_2}, \dfrac{\ln E_3}{\ln M_3}, \dfrac{\ln E_4}{\ln M_4} \sim p$

Alternatively, use least-squares fitting to find p

$$\underbrace{\ln \text{Rel Err}}_{y} \sim \underbrace{p}_{a}\,\underbrace{\ln M}_{x} + \underbrace{\ln C}_{b}$$

$y \sim ax + b$

$P = \text{polyfit}(x, y, n)$

$x = \ln M$, $y = \ln \text{Rel Err}$

for linear fun $n = 1$

coefficients of a fitted polynomial

$[a \;\; b]$, $a = P$, $b = \ln C$

By the same arguments as above:

$$(\lambda-\bar{\lambda})\iint_\Omega \phi\bar{\phi}\,dx\,dy=0$$

$$\phi=\underbrace{\phi_r+i\phi_i}_{\text{real-valued functions}} \Rightarrow \phi\bar{\phi}=(\phi_r+i\phi_i)(\phi_r-i\phi_i)=\phi_r^2+\phi_i^2>0$$

since $\phi\not\equiv 0$ as an e'function

$$\therefore \iint_\Omega \phi\bar{\phi}\,dx\,dy>0 \Rightarrow \lambda-\bar{\lambda}=0 \Rightarrow \lambda \text{ is real}$$

i.e. $\neq 0$

Gram-Schmidt process

In multidimensional case, it is possible to have several e'functions corresponding to the same e'value.

Suppose $\phi_1, \phi_2, \dots, \phi_n$ correspond to λ. These functions are linearly independent but not necessarily mutually orthogonal.

Q Can we make them orthogonal?

Key: any linear combination of $\phi_1, \phi_2, \dots, \phi_n$ is still an e'function corresponding to λ

Proof
let $\psi = \sum_{i=1}^{n} c_i \phi_i$ where $\nabla^2 \phi_i + \lambda \phi_i = 0$

Because ∇^2 is a linear operator $\nabla^2 \left(\sum_{i=1}^{n} c_i \phi_i \right) + \lambda \sum_{i=1}^{n} c_i \phi_i = 0$

$\therefore$ ψ is an e'function corresponding to λ.

Gram - Schmidt

$\phi_1, \phi_2, \dots, \phi_n \rightarrow \psi_1, \psi_2, \dots, \psi_n$

1. let $\psi_1 = \phi_1$
2. let $\psi_2 = \phi_2 + c_{11} \psi_1$ $\quad / \ \psi_1 \quad \iint_\Omega$

we need

$\cancel{\langle \psi_2, \psi_1 \rangle} \to 0$

$\langle \psi_2, \psi_1 \rangle = \langle \phi_2, \psi_1 \rangle + c_{11} \underbrace{\langle \psi_1, \psi_1 \rangle}_{\neq 0}$

Want: $\iint_\Omega \psi_1 \psi_2 \, dx\,dy = 0$

$$\Rightarrow \iint_\Omega \phi_2 \psi_1 \, dx\,dy + c_{11} \underbrace{\iint_\Omega \psi_1^2 \, dx\,dy}_{\neq 0} = 0$$

$$\therefore \quad C_{11} = -\frac{\iint_\Omega \phi_2 \psi_1 \, dx\, dy}{\iint_\Omega \psi_1^2 \, dx\, dy}$$

3. Let $\psi_3 = \phi_3 + C_{21}\psi_1 + C_{22}\psi_2 \qquad / \psi_1 \qquad / \cdot \psi_2$

We want

$$\iint_\Omega \psi_1 \psi_3 \, dx\, dy = 0$$

$$\iint_\Omega \psi_2 \psi_3 \, dx\, dy = 0$$

$$\iint_\Omega \phi_3 \psi_1 \, dx\, dy + C_{21} \underbrace{\iint_\Omega \psi_1^2 \, dx\, dy}_{} + C_{22} \cancelto{0}{\iint_\Omega \psi_1 \psi_2 \, dx\, dy} = 0$$

$$C_{21} = -\frac{\iint_\Omega \phi_3 \psi_1 \, dx\, dy}{\iint_\Omega \psi_1^2 \, dx\, dy}$$

Similarly

$$C_{32} = -\frac{\iint_\Omega \phi_3 \psi_2 \, dx\, dy}{\iint_\Omega \psi_2^2 \, dx\, dy}$$

In general,

$$\boxed{\psi_j = \phi_j - \sum_{i=1}^{j-1} \underbrace{\frac{\iint_\Omega \phi_j \psi_i \, dx\, dy}{\iint_\Omega \psi_i^2 \, dx\, dy}}_{C_{ij}} \cdot \psi_i}$$

$\therefore$ $\psi_1, \psi_2, \ldots, \psi_n$ are mutually orthogonal and correspond to the same e'value λ

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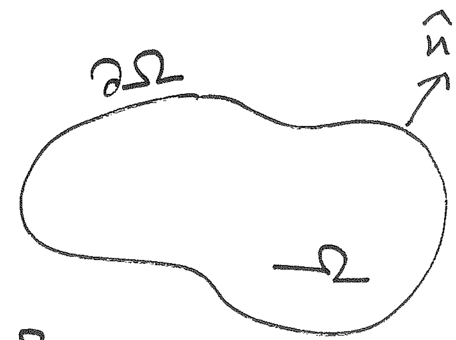
Rayleigh Quotient



$$\nabla^2\phi + \lambda\phi = 0 \quad \Big|\cdot\phi \quad \iint_\Omega$$

Multiply both sides by ϕ and integrate over Ω.

$$\Rightarrow \boxed{\lambda = -\frac{\iint_\Omega \phi\cdot\nabla^2\phi \, dx\, dy}{\iint_\Omega \phi^2 \, dx\, dy}}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

$$\vec{g} = (g_1, g_2)$$

$$\operatorname{div}\vec{g} = \nabla\cdot\vec{g} = \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y}$$

$$\nabla\cdot\nabla\phi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) =$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

$$\nabla\cdot(f\vec{g}) = \nabla f\cdot\vec{g} + f\,\nabla\cdot\vec{g}$$

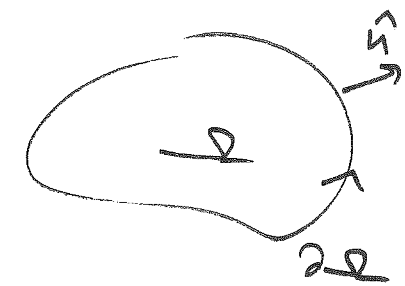
Let $f = \phi, \quad \vec{g} = \nabla\phi$

$$\Rightarrow \nabla\cdot(\phi\nabla\phi) = \underbrace{\nabla\phi\cdot\nabla\phi}_{|\nabla\phi|^2} + \phi\nabla^2\phi$$

$$\Rightarrow \quad \phi \nabla^2 \phi = \nabla \cdot (\phi \nabla \phi) - |\nabla \phi|^2$$

Hence,

$$\lambda = \frac{-\iint_{\Omega} \nabla \cdot (\phi \nabla \phi)\, dx\, dy + \iint_{\Omega} |\nabla \phi|^2 dx\, dy}{\iint_{\Omega} \phi^2 dx\, dy}$$



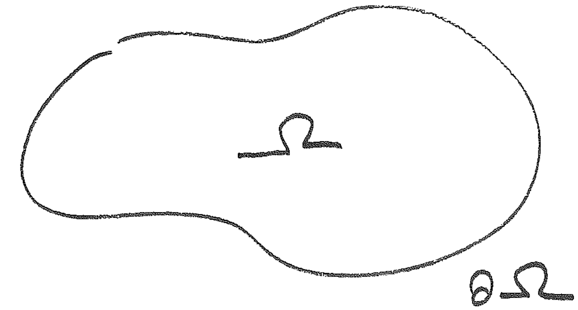
<u>Divergence Thm</u> : $\displaystyle\iint_{\Omega} \nabla \cdot \vec{A}\, dx\, dy = \oint_{\partial\Omega} \vec{A} \cdot \hat{n}\, ds$

Here: $\vec{A} = \phi \nabla \phi$

$$\therefore \quad \boxed{\lambda = \frac{-\oint_{\partial\Omega} \phi \nabla \phi \cdot \hat{n}\, dS + \iint_{\Omega} |\nabla \phi|^2 dx\, dy}{\iint_{\Omega} \phi^2 dx\, dy}}$$

<u>Rayleigh</u>
<u>Quotient</u>

Ex <u>Heat equation w/ Dirichlet BC</u>



$$u_t = K\nabla^2 u \quad \text{w/} \quad u\Big|_{\partial\Omega} = 0$$

$$u(x,y,t) = e^{-\lambda K t}\,\phi(x,y)$$

$\phi(x,y)$ satisfies Helmholtz eq^{n}:

$$\nabla^2\phi + \lambda\phi = 0 \quad \text{w/} \quad \phi\Big|_{\partial\Omega} = 0$$

Rayleigh Quotient:

$$\lambda = \frac{-\oint_{\partial\Omega} \cancelto{0}{\phi}\,\nabla\phi\cdot\hat{n}\,dS + \overbrace{\iint_{\Omega} |\nabla\phi|^2\,dx\,dy}^{\geq 0}}{\underbrace{\iint_{\Omega} \phi^2\,dx\,dy}_{>0}} \geq 0$$

Check if $\lambda = 0$ is an e'value.

$\lambda = 0$ is an e'value.

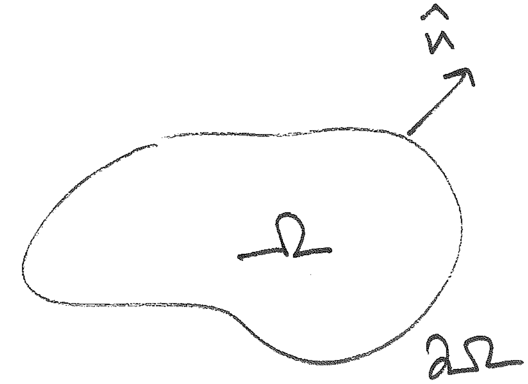
$$\lambda = \frac{\iint_\Omega |\nabla \phi|^2 \, dx\,dy}{\iint_\Omega \phi^2 \, dx\,dy} = 0 \Rightarrow \iint_\Omega \overbrace{|\nabla \phi|^2}^{\geq 0} \, dx\,dy = 0$$

$$\left.\begin{array}{l} \Rightarrow |\nabla \phi| = 0 \Rightarrow \phi = \text{const} \\ \text{but } \phi\big|_{\partial\Omega} = 0 \end{array}\right\} \Rightarrow \phi \equiv 0 : \text{ trivial solution}$$

Hence $\boxed{\lambda > 0}$

$\therefore$ $\boxed{\lim_{t \to \infty} u(x,y,t) = 0}$

<u>Ex</u> Heat equation w/ Neumann BC.



$$u_t = K\nabla^2 u \quad \text{w/} \quad \nabla u \cdot \hat{n}\Big|_{\partial\Omega} = 0$$

$$u(x,y,0) = f(x,y)$$

Rayleigh Quotient:

$$\lambda = \frac{-\oint_{\partial\Omega} \phi \underbrace{\nabla\phi\cdot\hat{n}}_{=0}\, dS + \iint_\Omega |\nabla\phi|^2\,dx\,dy}{\iint_\Omega \phi^2\,dx\,dy} = \frac{\overbrace{\iint_\Omega |\nabla\phi|^2\,dx\,dy}^{\geq 0}}{\underbrace{\iint_\Omega \phi^2\,dx\,dy}_{>0}} \geq 0$$

$$\lambda = 0 \Rightarrow \iint_\Omega |\nabla\phi|^2\,dx\,dy = 0 \Rightarrow |\nabla\phi| = 0 \Rightarrow \phi = \text{const}$$

$$\therefore \ \lambda \geq 0 \Rightarrow \lim_{t\to\infty} u(x,y,t) = u_\infty$$

From the integral conservation law:

$$\frac{d}{dt}\iint_{\Omega} c\rho u \, dx\, dy = \oint_{\partial\Omega} K_0 \underbrace{\nabla u \cdot \hat{n}}_{=0} \, dS$$

$$\therefore \quad \frac{d}{dt}\iint_{\Omega} c\rho u \, dx\, dy = 0 \quad \Rightarrow \quad \iint_{\Omega} c\rho u \, dx\, dy = \text{const}$$

$$\therefore \iint_{\Omega} c\rho \underbrace{u(x,y,0)}_{f(x,y)} \, dx\, dy = \iint_{\Omega} c\rho u(x,y,t) \, dx\, dy \quad \forall t \geq 0 \quad =$$

$$= \iint_{\Omega} c\rho \cdot \underset{\lim\limits_{t\to\infty} u(x,y,t)}{\underset{\shortparallel}{u_\infty}} \, dx\, dy$$

$f(x,y)$: initial temperature distribution

Denote by $A = \iint_\Omega dx\,dy$ = area of Ω

Assume $c = \text{const}$, $\rho = \text{const}$

$$\therefore \iint_\Omega u(x,y,t)\,dx\,dy = \iint_\Omega f(x,y)\,dx\,dy = u_\infty \cdot \underbrace{\iint_\Omega dx\,dy}_{A}$$

$$\boxed{u_\infty = \frac{\iint_\Omega f(x,y)\,dx\,dy}{\iint_\Omega dx\,dy}}$$

average initial temperature