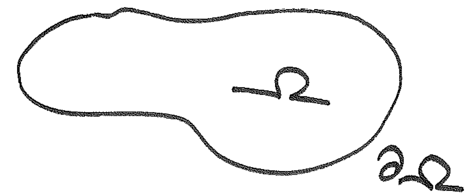
Lecture 42

Rayleigh Quotient



$$\nabla^2 \phi + \lambda \phi = 0 \quad / \cdot \phi \quad \iint_\Omega$$

Multiply both sides by ϕ and integrate.

$$\Rightarrow \quad \lambda = - \frac{\iint_\Omega \phi \nabla^2 \phi \, dx\, dy}{\iint_\Omega \phi^2 \, dx\, dy}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

$$\vec{g} = (g_1, g_2)$$

$$\nabla \cdot (f\vec{g}) = \nabla f \cdot \vec{g} + f \nabla \cdot \vec{g}$$

$$\operatorname{div} \vec{g} = \nabla \cdot \vec{g} = \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y}$$

Let $f = \phi$, $\vec{g} = \nabla \phi$

$$\nabla \cdot \nabla \phi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi$$

$$\nabla \cdot (\phi \nabla \phi) = \underbrace{\nabla \phi \cdot \nabla \phi}_{= |\nabla \phi|^2} + \phi \nabla^2 \phi$$

$$\Rightarrow \phi \nabla^2 \phi = \nabla \cdot (\phi \nabla \phi) - |\nabla \phi|^2$$

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$$\lambda = \frac{-\iint_\Omega \nabla\cdot(\phi\nabla\phi)\,dx\,dy + \iint_\Omega |\nabla\phi|^2\,dx\,dy}{\iint_\Omega \phi^2\,dx\,dy}$$

Divergence Thm: $\iint_\Omega \nabla\cdot\vec{A}\,dx\,dy = \oint_{\partial\Omega} \vec{A}\cdot\hat{n}\,dS$

$$\therefore \boxed{\lambda = \frac{-\oint_{\partial\Omega} \phi\nabla\phi\cdot\hat{n}\,dS + \iint_\Omega |\nabla\phi|^2\,dx\,dy}{\iint_\Omega \phi^2\,dx\,dy}}$$

Ex Heat equation

$u_t = k\nabla^2 u$ w/ $u\big|_{\partial\Omega} = 0$

$u(x,y,t) = e^{-\lambda k t}\,\phi(x,y)$

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$\nabla^2\phi + \lambda\phi = 0$ w/ $\phi\big|_{\partial\Omega} = 0$

Rayleigh Quotient:

$$\lambda = \frac{\iint_\Omega |\nabla\phi|^2\,dx\,dy}{\iint_\Omega \phi^2\,dx\,dy} \geq 0$$

> 0

$\nearrow$ $\lambda = 0 \Rightarrow \iint_\Omega |\nabla\phi|^2\,dx\,dy = 0 \Rightarrow |\nabla\phi| = 0 \Rightarrow \phi = \text{const}$

$\phi\big|_{\partial\Omega} = 0 \Rightarrow \phi \equiv 0$ ↯ $\Rightarrow \boxed{\lambda > 0}$

$$\therefore \boxed{\lim_{t\to\infty} u(x,y,t) = 0}$$

Ex $u_t = k\nabla^2 u$ w/ $\nabla u \cdot \hat{n} = 0$ on $\partial\Omega$

$$\Rightarrow \lambda = \frac{\iint_\Omega |\nabla\phi|^2\,dx\,dy}{\iint_\Omega \phi^2\,dx\,dy} \geq 0 \Rightarrow \lambda \geq 0$$

$$\lambda = 0 \Rightarrow |\nabla\phi| = 0 \Rightarrow \phi = \text{const}$$

$$\therefore \lambda \geq 0 \Rightarrow \lim_{t\to\infty} u(x,y,t) = u_\infty$$

From integral conservation law:

$$\frac{d}{dt}\iint_\Omega c\rho u\,dx\,dy = \oint_{\partial\Omega} k_0 \cancel{\nabla u \cdot \hat{n}}^{\,0}\,dS$$

$$\Rightarrow \frac{d}{dt}\iint_\Omega c\rho u\,dx\,dy = 0 \Rightarrow \iint_\Omega c\rho u\,dx\,dy = \text{const}$$

$$\Rightarrow \iint_\Omega c\rho \underbrace{u(x,y,0)}_{f(x,y)}\,dx\,dy = \iint_\Omega c\rho u(x,y,t)\,dx\,dy \quad \forall t \geq 0 \quad = \iint_\Omega c\rho \underbrace{u_\infty}_{\lim_{t\to\infty} u(x,y,t)}\,dx\,dy$$

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$f(x,y)$: initial temperature distribution

Denote by $A = \iint_\Omega dx\,dy$ = area of Ω

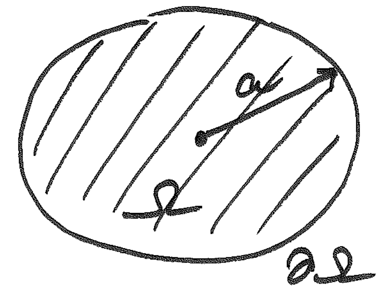
$c = \text{const}$ $\quad \rho = \text{const}$

$$\therefore \iint_\Omega u(x,y,t)\,dx\,dy = \underbrace{\iint_\Omega f(x,y)\,dx\,dy = u_\infty \iint_\Omega dx\,dy}$$

$$u_\infty = \frac{\iint_\Omega f(x,y)\,dx\,dy}{\iint_\Omega dx\,dy}$$: average initial temperature

Bessel Functions

Ex Vibrating Circular Membrane



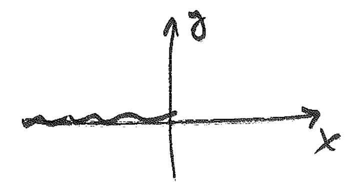
$$u_{tt} = c^2 \nabla^2 u = C^2 \frac{1}{r}\left((r u)_r + \frac{1}{r^2} u_{\theta\theta}\right)$$

$u(r, \theta, t)$

$\Omega = \{(r,\theta) : 0 < r < a, -\pi < \theta < \pi\}$

BCs: $u(a, \theta, t) = 0$

$|u(0, \theta, t)| < \infty$



ICs: $u(r, \theta, 0) = \alpha(r, \theta)$

$u_t(r, \theta, 0) = \beta(r, \theta)$

Separation of variables I:

$$u(r, \theta, t) = \phi(r, \theta)\, h(t)$$

$$\frac{\ddot{h}}{c^2 h} = \frac{1}{\phi}\left[\frac{1}{r}(r \phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta}\right] = -\lambda$$

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$\therefore \quad \ddot{h} + \lambda c^2 h = 0 \Rightarrow h(t) = c_1 \cos(\sqrt{\lambda} ct) + c_2 \sin(\sqrt{\lambda} ct)$

$\therefore \quad \frac{1}{r}(r\phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta} + \lambda\phi = 0 \qquad \phi(r,\theta)$

$\phi(a,\theta) = 0 \qquad |\phi(0,\theta)| < \infty$

Separation of variables II:

$$\phi(r,\theta) = f(r)\, g(\theta)$$

$$\frac{1}{r} g(\theta)\left(r f'(r)\right)' + \frac{1}{r^2} g''(\theta) f(r) + \lambda f(r) g(\theta) = 0$$

Divide by $\frac{1}{r^2} g(\theta) f(r)$.

$$\frac{r\left(r f'(r)\right)'}{f(r)} + \frac{g''(\theta)}{g(\theta)} + r^2 \lambda = 0$$

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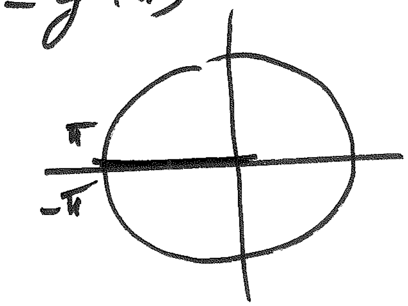
$$\therefore \quad \frac{r(r f'(r))'}{f(r)} + r^2 \lambda = -\frac{g''(\theta)}{g(\theta)} = \mu$$

$$\therefore \quad g''(\theta) + \mu g(\theta) = 0 \ \text{w/} \quad g(-\pi) = g(\pi), \quad g'(-\pi) = g'(\pi)$$



$$r(r f')' + (r^2 \lambda - \mu) f = 0$$

$$f(a) = 0 \quad \& \quad |f(0)| < \infty$$

I eigenvalue problem in θ-direction

$g(\theta) = a\cos(\sqrt{\mu}\,\theta) + b\sin(\sqrt{\mu}\,\theta)$, $\quad \sqrt{\mu}$: integer

$\mu_m = m^2$ for $m = 0, 1, 2 \ldots$ $\qquad \mu_m = \left(\frac{m\pi}{L}\right)^2$

$m=0 \qquad g_0(\theta) = 1$ $\qquad L = \pi$

$m>0 \qquad g_m(\theta) = \cos(m\theta)$ and $\sin(m\theta)$

II Eigenvalue problem in r-direction.

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0 \qquad \Big| \ \frac{1}{r}$$

$$f(a) = 0 \quad \text{and} \quad |f(0)| < \infty$$

Let $z = \sqrt{\lambda}\, r$.

$$\frac{d}{dr} = \frac{d}{dz}\cdot\frac{dz}{dr} = \sqrt{\lambda}\,\frac{d}{dz}$$

$\therefore$ $$\boxed{z^2 f''(z) + z f'(z) + (z^2 - m^2) f = 0}$$ Bessel's equation of order m

$$f(\sqrt{\lambda}\, a) = 0 \quad \& \quad |f(0)| < \infty$$

By Rayleigh Quotient, $\lambda > 0$. $0 < r < a \Rightarrow 0 < z < \sqrt{\lambda}\, a$

Note We can write this equation in Sturm-Liouville form:

$$\mathcal{L}(f) + \lambda \sigma f = 0 \qquad \mathcal{L} = \frac{d}{dr}\left(r \frac{d}{dr}\right) - \frac{m^2}{r}$$

$$\frac{d}{dx}\left[p \frac{d\phi}{dx}\right] + q\phi + \lambda\sigma\phi = 0$$

$$\Rightarrow x = r, \quad p(r) = r, \quad \sigma(r) = r, \quad q(r) = -\frac{m^2}{r}$$