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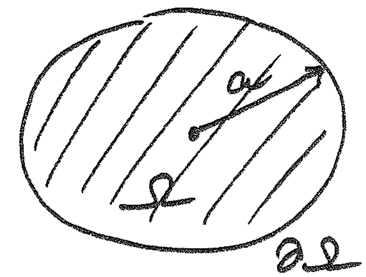
Lecture 42

Bessel Functions

Ex Vibrating Circular Membrane



$$u_{tt} = c^2 \nabla^2 u = c^2 \frac{1}{r}\left((ru)_r + \frac{1}{r^2} u_{\theta\theta}\right)$$

$u(r,\theta,t)$

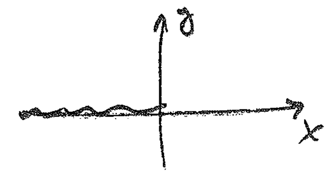
$\Omega = \{(r,\theta):\ 0<r<a,\ -\pi<\theta<\pi\}$

BCs: $u(a,\theta,t) = 0$

$|u(0,\theta,t)| < \infty$

ICs: $u(r,\theta,0) = \alpha(r,\theta)$

$u_t(r,\theta,0) = \beta(r,\theta)$



Separation of variables I:

$$u(r,\theta,t) = \phi(r,\theta)\, h(t)$$

$$\frac{\ddot{h}}{c^2 h} = \frac{1}{\phi}\left[\frac{1}{r}(r\phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta}\right] = -\lambda$$

E'value problem in r-direction:

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0 \qquad \Big/ \ \frac{1}{r}$$

$\mu = m^2$, $m = 0, 1, 2, \ldots$ BCs: $f(a) = 0$, $|f(0)| < \infty$

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right) f = 0$$

This is a Sturm-Liouville equation w/

$p(r) = r$, $q(r) = -\frac{m^2}{r}$, $\sigma(r) = r$

Note that this S.-L. problem is NOT regular because

1. BC at $r=0$ is NOT in a regular form.
2. $p(r)=0$, $\sigma(r)=0$ at $r=0$, which violates condition $p>0$, $\sigma>0$
3. $q(r) \to -\infty$ as $r \to 0$ $\Rightarrow$ $q(r)$ is not continuous at $r=0$

$\therefore \quad \ddot{h} + \lambda c^2 h = 0 \Rightarrow h(t) = c_1 \cos(\sqrt{\lambda} c t) + c_2 \sin(\sqrt{\lambda} c t)$

$\therefore \quad \frac{1}{r}(r \phi_r)_r + \frac{1}{r^2} \phi_{\theta\theta} + \lambda \phi = 0 \qquad \phi(r, \theta)$

$\phi(a, \theta) = 0 \qquad |\phi(0, \theta)| < \infty$

Separation of variables II:

$$\phi(r, \theta) = f(r) g(\theta)$$

$$\frac{1}{r} g(\theta) (r f'(r))' + \frac{1}{r^2} g''(\theta) f(r) + \lambda f(r) g(\theta) = 0$$

Divide by $\frac{1}{r^2} g(\theta) f(r)$.

$$\frac{r (r f'(r))'}{f(r)} + \frac{g''(\theta)}{g(\theta)} + r^2 \lambda = 0$$

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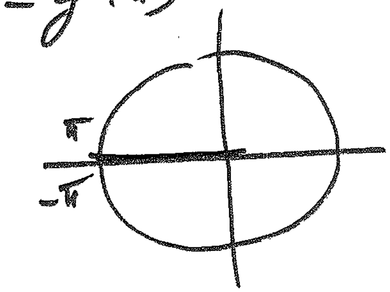
$$\therefore \quad \frac{r(r f'(r))'}{f(r)} + r^2 \lambda = -\frac{g''(\theta)}{g(\theta)} = \mu$$

$$\therefore \quad g''(\theta) + \mu g(\theta) = 0 \quad w/ \quad g(-\pi) = g(\pi), \quad g'(-\pi) = g'(\pi)$$



$$r(r f')' + (r^2 \lambda - \mu) f = 0$$

$$f(a) = 0 \quad \& \quad |f(0)| < \infty$$

I Eigenvalue problem in θ-direction

$$g(\theta) = a \cos(\sqrt{\mu}\,\theta) + b \sin(\sqrt{\mu}\,\theta), \qquad \sqrt{\mu} : \text{integer}$$

$$\mu_m = m^2 \quad \text{for} \quad m = 0, 1, 2 \ldots$$

$$\mu_m = \left(\frac{m\pi}{L}\right)^2$$

$$L = \pi$$

$m=0 \qquad g_0(\theta) = 1$

$m>0 \qquad g_m(\theta) = \cos(m\theta)$ and $\sin(m\theta)$

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II Eigenvalue problem in r-direction.

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0 \qquad \Big| \ \frac{1}{r}$$

$$f(a) = 0 \quad \text{and} \quad |f(0)| < \infty$$

Let $z = \sqrt{\lambda}\, r$.

$$\frac{d}{dr} = \frac{d}{dz}\cdot\frac{dz}{dr} = \sqrt{\lambda}\frac{d}{dz} \qquad \frac{d^2}{dr^2} = \lambda\frac{d^2}{dz^2}$$

$$\therefore \quad \boxed{z^2 f''(z) + z f'(z) + (z^2 - m^2) f = 0} \qquad \text{Bessel's equation of order } m$$

$$f(\sqrt{\lambda}\, a) = 0 \quad \& \quad |f(0)| < \infty$$

By Rayleigh Quotient, $\lambda > 0$. $\quad 0 < r < a \Rightarrow 0 < z < \sqrt{\lambda}\, a$

Note We can write this equation in Sturm-Liouville form:

$\mathcal{L}(f) + \lambda\sigma f = 0$ $\quad\quad$ $\mathcal{L} = \frac{d}{dr}\left(r \frac{d}{dr}\right) - \frac{m^2}{r}$

$$\frac{d}{dx}\left[p \frac{d\phi}{dx}\right] + q\phi + \lambda\sigma\phi = 0$$

$\Rightarrow x = r, \quad p(r) = r, \quad \sigma(r) = r, \quad q(r) = -\frac{m^2}{r}$

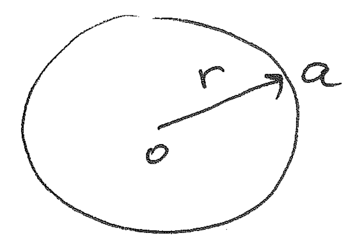
It turns out, however, that the S.-L. theorem still applies to this problem:

1. For each m, there are ∞ many e'values λ_{nm}, $m=0,1,\ldots$ $n=1,2,3,\ldots$
2. $f_{nm}(r)$ is e'function corresponding to e'value λ_{nm}.
3. e'functions are orthogonal wrt weight r:

$$\int_0^a f_{n_1,m}(r)\, f_{n_2,m}(r) \cdot r\,dr = 0 \qquad n_1 \neq n_2$$

$0 \leq r \leq a$

$\sigma(r) = r$



Bessel's Equation:

$$\boxed{z^2 f'' + z f' + (z^2 - m^2) f = 0}$$

$$\rightarrow f'' + \frac{1}{z} f' + \left(1 - \frac{m^2}{z^2}\right) f = 0 \; : \qquad z=0 \text{ is a singular point}$$

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Behaviour near $z=0$: $z^2 f \approx 0$ near $z=0$

<u>Note</u>: zf' and $z^2 f''$ are not necessarily small near $z=0$ because derivatives of f might be huge.

$\therefore \quad z^2 f'' + z f' - m^2 f \approx 0$ near $z=0$: equipotential / equidimensional eq^n

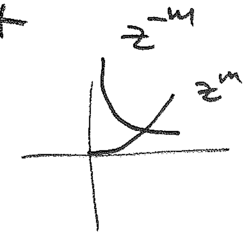
Let $f(z) \approx z^s \Rightarrow f' \approx s z^{s-1}, \quad f'' \approx s(s-1) z^{s-2}$

$$z^2 \cdot s(s-1) z^{s-2} + z \cdot s z^{s-1} - m^2 z^s = 0$$

$z^s [\, s(s-1) + s - m^2] = 0 \quad \Rightarrow \quad s(s-1) + s - m^2 = 0$: indicial eq^n

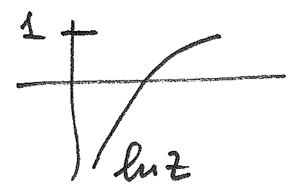
$\Rightarrow s^2 = m^2 \Rightarrow \quad s = \pm m$ if $m \neq 0$

$s = 0$ if $m=0$: repeated root



$\therefore$ if $m>0$: $f \approx C_1 z^m + C_2 z^{-m}$

if $m=0$: $f \approx C_1 + C_2 \ln z$



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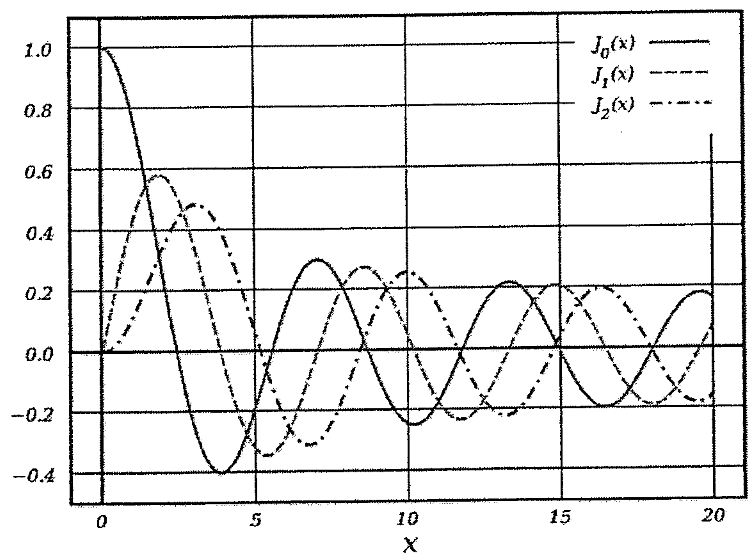
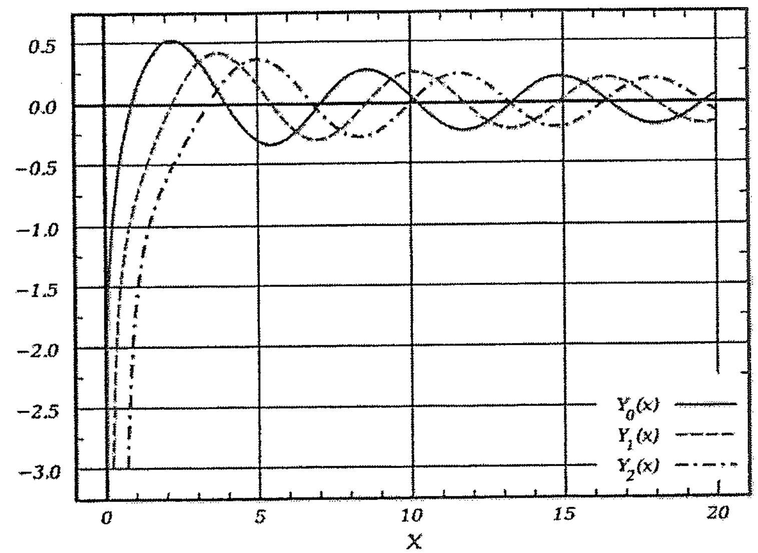
Graphs of the Bessel functions of the 1[st] and 2[nd] kind for m=0, 1 and 2





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Bessel Functions:

The exact solution to Bessel's equation can be written in terms of $J_m(z)$ (Bessel function of the 1st kind of order m) and $Y_m(z)$ (Bessel function of the 2nd kind of order m):

$$\boxed{f(z) = C_1 J_m(z) + C_2 Y_m(z)}$$

Note

1. $J_m(z)$ is well-behaved at $z=0$:

$$J_m(z) \sim \begin{cases} 1 & m=0 \\ \frac{1}{2^m m!} z^m, & m>0 \end{cases} \quad \text{as } z \to 0$$

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2. $Y_m(z)$ blows up at $z=0$:

$$Y_m(z) \sim \begin{cases} \frac{2}{\pi} \ln z & m=0 \\ -\frac{2^m (m-1)!}{\pi} z^{-m} & m>0 \end{cases} \quad \text{as } z \to 0$$

One cannot write the Bessel's functions in terms of elementary functions (exp, cos, sin, ln), but it is possible to get an <u>infinite series representation</u> and some <u>asymptotic formulas</u>.

Back to Bessel's problem:

$$z^2 f'' + z f' + (z^2 - m^2) f = 0 \qquad z = \sqrt{\lambda}\, r$$

w/ $f(\sqrt{\lambda}\, a) = 0 \qquad |f(0)| < \infty$

$$f(z) = C_1 J_m(z) + C_2 Y_m(z)$$

$$|f(0)| < \infty \Rightarrow C_2 = 0$$

$$z = \sqrt{\lambda}\, r$$

$$f(z) = C_1 J_m(z)$$

$$f(\sqrt{\lambda} a) = 0 \Rightarrow \underset{\neq 0}{C_1} J_m(\sqrt{\lambda} a) = 0 \Rightarrow J_m(\sqrt{\lambda} a) = 0$$

$\therefore$ e'values are related to the roots of the Bessel function of the 1st kind

Let z_{mn} be the n^{th} root of $J_m(z)$.

$$\Rightarrow \sqrt{\lambda} a = z_{mn} \Rightarrow \boxed{\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2 : \text{ e'values}}$$

For each $m = 0, 1, 2, \ldots$ there are ∞ many roots $n = 1, 2, \ldots$

$$\boxed{\text{e'functions: } f(z) = J_m\left(\frac{z_{mn}}{a} r\right) \text{ for } m = 0, 1, 2, \ldots \;\; n = 1, 2, \ldots}$$

<u>Orthogonality</u>:

$$\int_0^a J_m\left(\frac{z_{mp}}{a} r\right) J_m\left(\frac{z_{mq}}{a} r\right) r\, dr = 0 \qquad p \neq q$$

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Completeness: any piecewise smooth function $u(r)$ can be written as

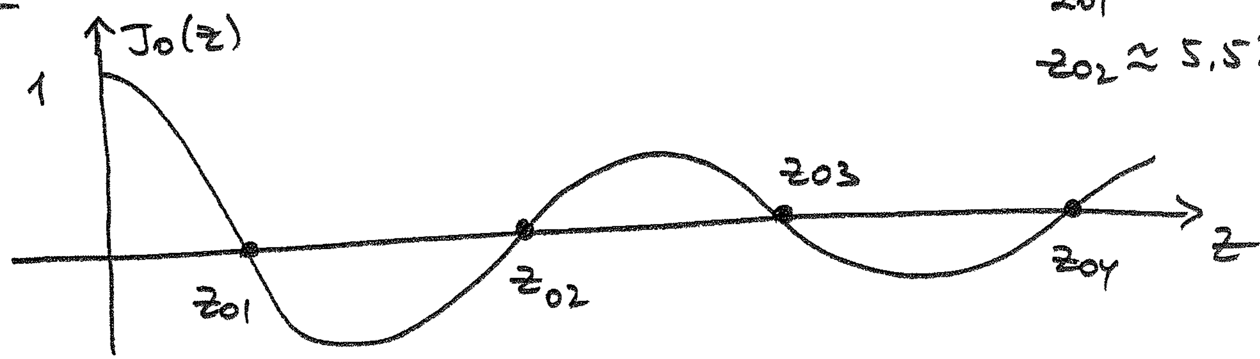
$$u(r) = \sum_{n=1}^{\infty} a_n J_m\left(z_{mn} \frac{r}{a}\right):$$

Fourier - Bessel series

where using orthogonality:

$$a_n = \frac{\int_0^a u(r) J_m\left(z_{mn} \frac{r}{a}\right) r\, dr}{\int_0^a J_m^2\left(z_{mn} \frac{r}{a}\right) r\, dr}$$

Ex $J_0(z)$ Bessel function of 1st kind of order 0



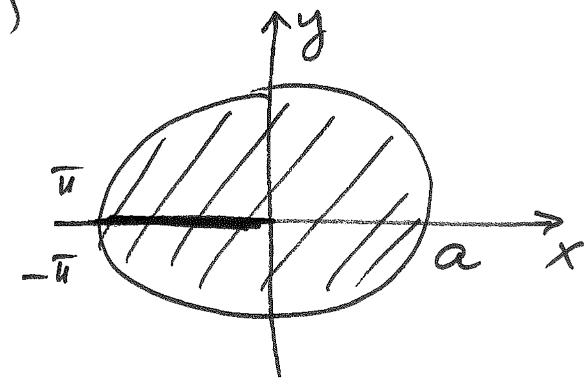
$z_{01} \approx 2.40483\ldots$

$z_{02} \approx 5.52008\ldots$

$z_{03} \approx 8.65373\ldots$

$z_{04} \approx 11.79153\ldots$

Ex Circular membrane (Cont'd)



$$u_{tt} = c^2 \nabla^2 u$$

$$|u(0, \theta, t)| < \infty, \qquad u(a, \theta, t) = 0$$

$$u(r, \theta, 0) = \alpha(r, \theta)$$

$$u_t(r, \theta, 0) = 0$$

$$\therefore \quad u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_m\left(\frac{z_{mn}}{a} r\right) \cos(m\theta) \cos\left(c \frac{z_{mn}}{a} t\right) +$$

$$+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{mn} J_m\left(\frac{z_{mn}}{a} r\right) \sin(m\theta) \cos\left(c \frac{z_{mn}}{a} t\right)$$

$$A_{0n} = \frac{\int_{-\pi}^{\pi} \int_{0}^{a} \alpha(r,\theta)\, J_0\left(\frac{z_{0n}}{a} r\right) r\, dr\, d\theta}{2\pi \int_{0}^{a} J_0^2\left(\frac{z_{0n}}{a} r\right) r\, dr}$$

$$A_{mn} = \frac{\int_{-\pi}^{\pi} \int_{0}^{a} \alpha(r,\theta)\, J_m\left(\frac{z_{mn}}{a} r\right) \cos(m\theta) \cdot r\, dr\, d\theta}{\pi \int_{0}^{a} J_m^2\left(\frac{z_{mn}}{a} r\right) r\, dr}$$

$$B_{mn} = \frac{\int_{-\pi}^{\pi} \int_{0}^{a} \alpha(r,\theta)\, J_m\left(\frac{z_{mn}}{a} r\right) \sin(m\theta) \cdot r\, dr\, d\theta}{\pi \int_{0}^{a} J_m^2\left(\frac{z_{mn}}{a} r\right) r\, dr}$$

Circularly symmetric case

Q What if the initial condition is independent of θ?

A The full solution must remain independent of θ.

Why? Because the variables r, θ, t are separated (θ-dependent problem is decoupled) and BCs are <u>independent</u> of θ.

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$$\Rightarrow \quad u_{tt} = c^2 \cdot \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad u = u(r,t)$$

$$u(a,t)=0 \qquad |u(0,t)|<\infty$$

$$u(r,0)=\alpha(r) \qquad u_t(r,0)=\beta(r)$$

sep. of variables: $u(r,t) = \phi(r)\,h(t)$

$$\frac{\ddot{h}}{c^2 h} = \frac{1}{r\phi}(r\phi')' = -\lambda$$

$$\therefore \quad h^2 + \lambda c^2 h = 0 \quad \Rightarrow \quad h(t) = c_1\cos(\sqrt{\lambda}ct) + c_2\sin(\sqrt{\lambda}ct)$$

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0$$

$$\phi(a)=0 \qquad |\phi(0)|<\infty$$

$$r\frac{d^2\phi}{dr^2} + \frac{d\phi}{dr} + \lambda r\phi = 0$$

Let $z = \sqrt{\lambda}\, r$

$$\frac{d}{dr} = \frac{d}{dz}\cdot\frac{dz}{dr} = \sqrt{\lambda}\frac{d}{dz} \qquad \frac{d^2}{dr^2} = \lambda\frac{d^2}{dz^2}$$

$$r.\lambda \frac{d^2\phi}{dz^2} + \sqrt{\lambda}\frac{d\phi}{dz} + \lambda r \phi = 0 \quad / r$$

$$\underbrace{r^2\lambda}_{z^2} \frac{d^2\phi}{dz^2} + \underbrace{r\sqrt{\lambda}}_{z}\frac{d\phi}{dz} + \underbrace{\lambda r^2}_{z^2}\phi = 0$$

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + z^2\phi = 0$$: Bessel equation of order 0

Recall

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + (z^2 - m^2)\phi = 0$$ Bessel equation of order m

$$\therefore \phi(z) = C_1 J_0(z) + C_2 Y_0(z)$$

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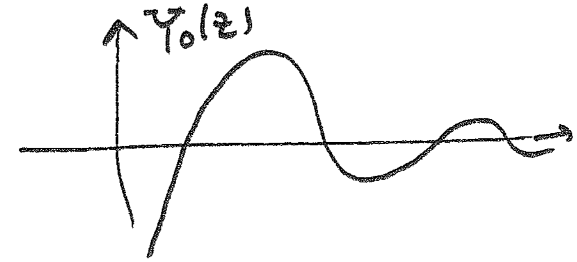
BC : $|\phi(0)| < \infty \Rightarrow C_2 = 0$ since $Y_0(z)$ is unbounded at $z=0$



$z = \sqrt{\lambda}\, r$

$\Rightarrow \phi(z) = C_1 J_0(z)$

BC: $\phi(a) = 0 \Rightarrow \underset{\neq 0}{C_1} J_0(\sqrt{\lambda}\, a) = 0$

$\Rightarrow \sqrt{\lambda}\, a = z_{0n}$: roots of $J_0(z)$, Bessel function of 1st kind of order 0

$n = 1, 2, \ldots$

$\sqrt{\lambda} = \dfrac{z_{0n}}{a}$

$\lambda = \left(\dfrac{z_{0n}}{a}\right)^2$: e'values

$n = 1, 2, \ldots$

$\phi_n(r) = J_0\left(\dfrac{z_{0n}}{a} r\right)$: e'functions

Since $h(t) = C_1 \cos\left(\underbrace{\frac{z_{0n}}{a}}_{\sqrt{\lambda}} ct\right) + C_2 \sin\left(\underbrace{\frac{z_{0n}}{a}}_{\sqrt{\lambda}} ct\right)$

the solution is

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} a_n J_0\left(\frac{z_{0n}}{a} r\right) \cos\left(\frac{z_{0n}}{a} ct\right) + b_n J_0\left(\frac{z_{0n}}{a} r\right) \sin\left(\frac{z_{0n}}{a} ct\right)}$$

ICs: at $t=0$ $\quad \alpha(r) = \sum_{n=1}^{\infty} a_n J_0\left(\frac{z_{0n}}{a} r\right)$

$$\therefore \quad a_n = \frac{\int_0^a \alpha(r) J_0\left(\frac{z_{0n}}{a} r\right) r\,dr}{\int_0^a J_0^2\left(\frac{z_{0n}}{a} r\right) r\,dr}$$

$$b_n = \frac{a}{z_{0n} c} \frac{\int_0^a \beta(r) J_0\left(\frac{z_{0n}}{a} r\right) r\,dr}{\int_0^a J_0^2\left(\frac{z_{0n}}{a} r\right) r\,dr}$$