

Eigenvalue problem in r -direction:

$$r \frac{d}{dr} \left(r \frac{df}{dr} \right) + (2r^2 - \mu) f = 0 \quad | \quad \frac{1}{r}$$

BCs: $f(a) = 0, \quad |f(0)| < \infty$
 $\mu = m^2, \quad m = 0, 1, 2, \dots$

$$\frac{d}{dr} \left(r \frac{df}{dr} \right) + \left(2r - \frac{m^2}{r} \right) f = 0$$

This is a Sturm-Liouville equation w/

$$p(r) = r, \quad q(r) = -\frac{m^2}{r}, \quad b(r) = r$$

Note that this S.-L. problem is NOT regular because

1. BC at $r=0$ is NOT in a regular form.

2. $p(r) = 0, \quad b(r) = 0$ at $r=0$, which violates condition

$$p > 0, \quad b > 0$$

3. $q(r) \rightarrow -\infty$ as $r \rightarrow 0 \Rightarrow q(r)$ is not continuous at $r=0$

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It turns out, however, that the S.-L. theorem still applies to this problem:

1. For each m , there are ∞ many e 'values λ_{nm} , $n=1, 2, 3, \dots$
2. $f_{nm}(r)$ is e 'function corresponding to e 'value λ_{nm} .
3. e 'functions are orthogonal wrt weight r :

$$\int_0^a f_{n_1, m}(r) f_{n_2, m}(r) \cdot r \, dr = 0 \quad n_1 \neq n_2$$

Bessel's Equation:

$$z^2 f'' + z f' + (z^2 - m^2) f = 0$$

$$\rightarrow f'' + \frac{1}{z} f' + \left(1 - \frac{m^2}{z^2}\right) f = 0 \quad : \quad z=0 \text{ is a singular point}$$

Behaviour near $z=0$: $z^2 f \approx 0$ near $z=0$

Note: $z f'$ and $z^2 f''$ are not necessarily small near $z=0$ because derivatives of f might be huge.

$\therefore z^2 f'' + z f' - m^2 f \approx 0$ near $z=0$: equipotential / equidimensional eg $z^{\frac{1}{2}}$

Let $f(z) \approx z^s \Rightarrow f' \approx s z^{s-1}, f'' \approx s(s-1) z^{s-2}$

$$z^2 \cdot s(s-1) z^{s-2} + z \cdot s z^{s-1} - m^2 z^s = 0$$

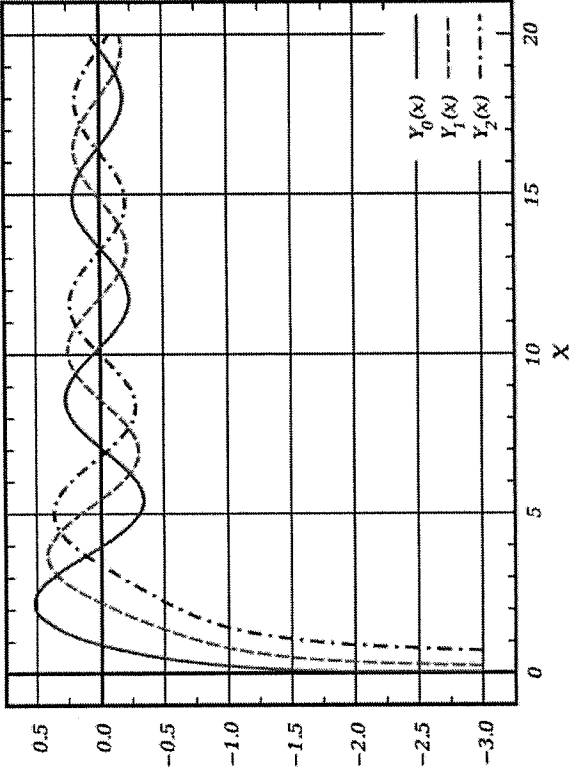
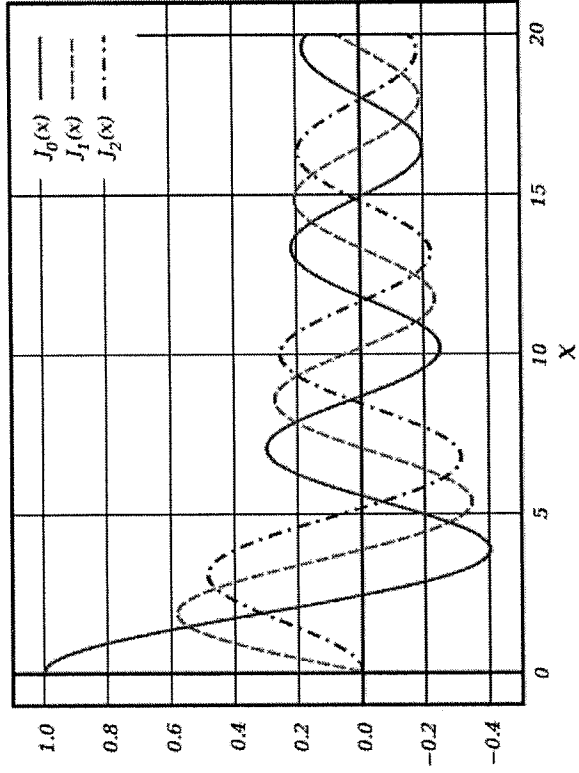
$$z^s [s(s-1) + s - m^2] = 0 \Rightarrow s(s-1) + s - m^2 = 0 : \text{indicial eq}^{\frac{1}{2}}$$

$$\Rightarrow s^2 = m^2 \Rightarrow \begin{array}{l} s = \pm m \quad \text{if } m \neq 0 \\ s = 0 \quad \text{if } m = 0 : \text{repeated root} \end{array}$$

$$\therefore \text{if } m > 0 : f \approx C_1 z^m + C_2 z^{-m}$$

$$\text{if } m = 0 : f \approx C_1 + C_2 \ln z$$

Graphs of the Bessel functions of the 1st and 2nd kind for $m=0, 1$ and 2



Bessel Functions:

The exact solution to Bessel's equation can be written in terms of $J_m(z)$ (Bessel function of the 1st kind of order m) and $Y_m(z)$ (Bessel function of the 2nd kind of order m):

$$f(z) = C_1 J_m(z) + C_2 Y_m(z)$$

Note

1. $J_m(z)$ is well-behaved at $z=0$:

$$J_m(z) \sim \begin{cases} 1 & m=0 \\ \frac{1}{2^m m!} z^m & m>0 \end{cases} \quad \text{as } z \rightarrow 0$$

2. $Y_m(z)$ blows up at $z=0$:

$$Y_m(z) \sim \begin{cases} \frac{2}{\pi} \ln z & m=0 \\ -\frac{2^{m(m-1)!}}{\pi} z^{-m} & m>0 \end{cases} \quad \text{as } z \rightarrow 0$$

One cannot write the Bessel's functions in terms of elementary functions (exp, cos, sin, ln), but it is possible to get an infinite series representation and some asymptotic formulas.

Back to Bessel's problem:

$$z^2 f'' + z f' + (z^2 - m^2) f = 0$$

$$\text{w/ } f(\sqrt{a}) = 0 \quad |f(0)| < \infty$$

$$f(z) = C_1 J_m(z) + C_2 Y_m(z)$$

$$|f(0)| < \infty \Rightarrow C_2 = 0$$

$$z = \sqrt{a} r$$

$$f(z) = C_1 J_m(z)$$

$$f(\sqrt{a}) = 0 \Rightarrow$$

$$C_1 J_m(\sqrt{a}) = 0 \Rightarrow J_m(\sqrt{a}) = 0$$

\therefore e'values are related to the roots of the Bessel function of the 1st kind

Let z_{mn} be the n^{th} root of $J_m(z)$.

$$z_{mn} = \left(\frac{z_{mn}}{a} \right)^2 : \text{e'values}$$

$$\Rightarrow \sqrt{a} z = z_{mn} \Rightarrow$$

For each $m = 0, 1, 2, \dots$ there are ∞ many roots $n = 1, 2, \dots$

e'functions: $f(z) = J_m\left(\frac{z_{mn}}{a} r\right)$ for $m = 0, 1, 2, \dots$
 $n = 1, 2, \dots$

Orthogonality: $\int_0^a J_m\left(\frac{z_{mp}}{a} r\right) J_m\left(\frac{z_{mq}}{a} r\right) r dr = 0$
 $p \neq q$

Completeness: any piecewise smooth function $u(r)$ can be written as

$$u(r) = \sum_{n=1}^{\infty} a_n J_m \left(z_{mn} \frac{r}{a} \right): \quad \text{Fourier - Bessel series}$$

where using orthogonality:

$$a_n = \frac{\int_0^a u(r) J_m \left(z_{mn} \frac{r}{a} \right) r dr}{\int_0^a J_m^2 \left(z_{mn} \frac{r}{a} \right) r dr}$$

Ex $J_0(z)$ Bessel function of 1st kind of order 0

$$z_{01} \approx 2.40483 \dots$$

$$z_{02} \approx 5.52008 \dots$$

$$z_{03} \approx 8.65373 \dots$$

$$z_{04} \approx 11.79153 \dots$$

