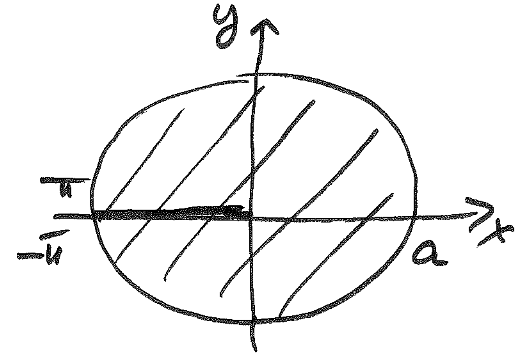
Lecture 44

Ex Circular membrane



$u_{tt} = c^2 \nabla^2 u$

$|u(0,\theta,t)| < \infty \qquad u(a,\theta,t) = 0$

$u(r,\theta,0) = \alpha(r,\theta)$

$u_t(r,\theta,0) = 0$

$$\therefore \; u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m\left(\frac{z_{mn}}{a} r\right)\cos(m\theta)\cos\left(c\frac{z_{mn}}{a}t\right) +$$

$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m\left(\frac{z_{mn}}{a} r\right)\sin(m\theta)\cos\left(c\frac{z_{mn}}{a}t\right)$$

$$A_{0n} = \int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta) J_0\left(\frac{z_{0n}}{a} r\right) r\,dr\,d\theta \Big/ 2\pi \int_0^a J_0^2\left(\frac{z_{0n}}{a} r\right) r\,dr$$

$$A_{mn} = \int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\, J_m\!\left(\frac{z_{mn}}{a} r\right) \cos(m\theta)\, r\, dr\, d\theta \Big/ \pi \int_0^a J_m^2\!\left(\frac{z_{mn}}{a} r\right) r\, dr$$

$$B_{mn} = \int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\, J_m\!\left(\frac{z_{mn}}{a} r\right) \sin(m\theta)\, r\, dr\, d\theta \Big/ \pi \int_0^a J_m^2\!\left(\frac{z_{mn}}{a} r\right) r\, dr$$

<u>Circularly symmetric case</u>

Q: What if the initial condition is independent of θ?

A. The full solution must remain independent of θ.

Why? Because the variables r, θ, t are <u>separated</u> and BCs are <u>independent</u> of θ.

$\Rightarrow \quad u_{tt} = c^2 \cdot \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) \qquad u = u(r,t)$

$u(a,t) = 0 \qquad |u(0,t)| < \infty$

$u(r,0) = \alpha(r) \qquad u_t(r,0) = \beta(r)$

sep. of variables: $u(r,t) = \phi(r)\, h(t)$

$\frac{\ddot{h}}{c^2 h} = \frac{1}{r\phi}(r\phi')' = -\lambda$

$\therefore \quad h^2 + \lambda c^2 h = 0 \quad \Rightarrow \quad h(t) = c_1 \cos(\sqrt{\lambda}\, ct) + c_2 \sin(\sqrt{\lambda}\, ct)$

$\frac{d}{dr}\left(r \frac{d\phi}{dr}\right) + \lambda r \phi = 0$

$\phi(a) = 0 \qquad |\phi(0)| < \infty$

$r \frac{d^2\phi}{dr^2} + \frac{d\phi}{dr} + \lambda r \phi = 0$

Let $z = \sqrt{\lambda}\, r$

$$\frac{d}{dr} = \frac{d}{dz} \cdot \frac{dz}{dr} = \sqrt{\lambda} \frac{d}{dz} \qquad \frac{d^2}{dr^2} = \lambda \frac{d^2}{dz^2}$$

$$r \cdot \lambda \frac{d^2\phi}{dz^2} + \sqrt{\lambda} \frac{d\phi}{dz} + \lambda r \phi = 0 \quad / r$$

$$\underbrace{r^2\lambda}_{z^2} \frac{d^2\phi}{dz^2} + \underbrace{r\sqrt{\lambda}}_{z} \frac{d\phi}{dz} + \underbrace{\lambda r^2}_{z^2}\phi = 0$$

$$z^2 \frac{d^2\phi}{dz^2} + z \frac{d\phi}{dz} + z^2 \phi = 0$$: Bessel equation of order 0

Recall

$$z^2 \frac{d^2\phi}{dz^2} + z \frac{d\phi}{dz} + (z^2 - m^2)\phi = 0$$ Bessel equation of order m

$$\therefore \phi(z) = C_1 J_0(z) + C_2 Y_0(z)$$

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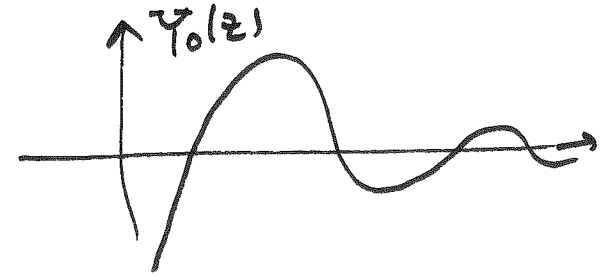
BC : $|\phi(0)| < \infty \Rightarrow C_2 = 0$ since $Y_0(z)$ is unbounded at $z=0$



$z = \sqrt{\lambda}\, r$

$\Rightarrow \phi(z) = C_1 J_0(z)$

BC: $\phi(a) = 0 \Rightarrow \underset{\neq 0}{C_1} J_0(\sqrt{\lambda}\, a) = 0$

$\Rightarrow \sqrt{\lambda}\, a = z_{0n}$: roots of $J_0(z)$, Bessel function of 1st kind of order 0

$n = 1, 2, \ldots$

$\sqrt{\lambda} = \dfrac{z_{0n}}{a}$

$\lambda = \left(\dfrac{z_{0n}}{a}\right)^2$: e'values

$n = 1, 2, \ldots$

$\phi_n(r) = J_0\left(\dfrac{z_{0n}}{a} r\right)$: e'functions

Since $h(t) = C_1 \cos\left(\underbrace{\frac{z_{0n}}{a}}_{\sqrt{\lambda}} ct\right) + C_2 \sin\left(\underbrace{\frac{z_{0n}}{a}}_{\sqrt{\lambda}} ct\right)$

the solution is

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} a_n J_0\left(\frac{z_{0n}}{a} r\right) \cos\left(\frac{z_{0n}}{a} ct\right) + b_n J_0\left(\frac{z_{0n}}{a} r\right) \sin\left(\frac{z_{0n}}{a} ct\right)}$$

ICs: at $t=0$ $\quad \alpha(r) = \sum_{n=1}^{\infty} a_n J_0\left(\frac{z_{0n}}{a} r\right)$

$$\therefore \quad a_n = \frac{\int_0^a \alpha(r) J_0\left(\frac{z_{0n}}{a} r\right) r\, dr}{\int_0^a J_0^2\left(\frac{z_{0n}}{a} r\right) r\, dr}$$

$$b_n = \frac{a}{z_{0n} c} \frac{\int_0^a \beta(r) J_0\left(\frac{z_{0n}}{a} r\right) r\, dr}{\int_0^a J_0^2\left(\frac{z_{0n}}{a} r\right) r\, dr}$$

Q What if (for more general problem) the initial condition is independent of r?

A Unless $u(r,\theta,0)=0$, $u_t(r,\theta,0)=0$, the solution will still develop r-dependence for $t>0$ because of BC (BC is at $r=a$, ~~not~~ independent of r):

$$u(a,\theta,t)=0 \quad : \text{ depends on } r$$

Note The circularly symmetric solution is the $m=0$ part of more general problem.

More on Bessel functions

Recall

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0 \quad : \quad \text{Bessel equation of order } m$$

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Asymptotics I : small z $(z \to 0)$

$$\Rightarrow z^2 f'' + z f' - m^2 f = 0 \qquad (\text{drop } z^2 f \text{ term})$$

let $f = z^p$

$$z^2 \cdot p(p-1) z^{p-2} + z \cdot p z^{p-1} - m^2 z^p = 0$$

indicial eq^{n}: $p(p-1) + p - m^2 = 0 \Rightarrow p^2 = m^2 \Rightarrow p = \pm m$

$$\therefore \quad f(z) = C_1 z^{m} + C_2 z^{-m} \qquad \text{if } m \neq 0$$

$$f(z) = C_1 + C_2 \ln z \qquad \text{if } m = 0$$

1st kind: $J_0(z) \approx 1$, $J_1(z) \approx \frac{1}{2} z$, $J_2(z) \approx \frac{1}{8} z^2$, ...

2nd kind: $Y_0(z) \approx \frac{2}{\pi} \ln z$, $Y_1(z) \approx -\frac{2}{\pi} z^{-1}$, $Y_2(z) \approx -\frac{4}{\pi} z^{-2}$, ...

Asymptotics II : large z ($z \to \infty$)

$$z^2 f'' + z f' + (z^2 - m^2) f = 0 \quad \Big| \ \frac{1}{z^2}$$

Recall $z = \sqrt{\lambda}\, r \Rightarrow$ large z implies large λ since $0 \le r \le a$.

$$f'' + \underbrace{\frac{1}{z}}_{c} f' + \underbrace{\left(1 - \frac{m^2}{z^2}\right)}_{k} f = 0$$

compare w/ mass-spring system w/ damping

$x(t)$: displacement

$$m\ddot{x} + c\dot{x} + kx = 0$$

looks like mass-spring system w/ damping w/

$c \sim \frac{1}{z}$: small $\quad c \to 0$ as $z \to \infty$

$k \sim \left(1 - \frac{m^2}{z^2}\right) \sim 1$ as $z \to \infty$

assume $x = e^{pt}$

$$m p^2 e^{pt} + c p e^{pt} + k e^{pt} = 0$$

$$m p^2 + c p + k = 0$$

$$p = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} =$$

$$= -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}$$

since $c \sim 0$, $\frac{k}{m} > 0 \Rightarrow \frac{c^2}{4m^2} - \frac{k}{m} < 0$: underdamped case

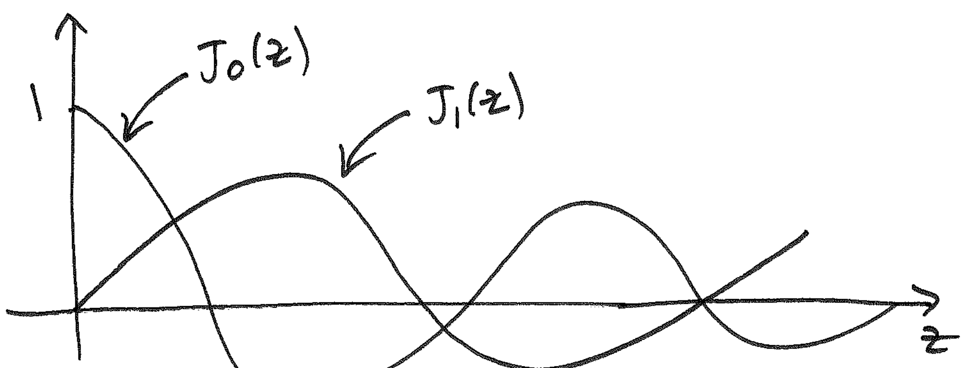
$$p = -\frac{c}{2m} \pm i\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

$$x(t) = e^{-\frac{c}{2m}t}\left[C_1 \cos\left(\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}\, t\right) + C_2 \sin\left(\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}\, t\right)\right]$$

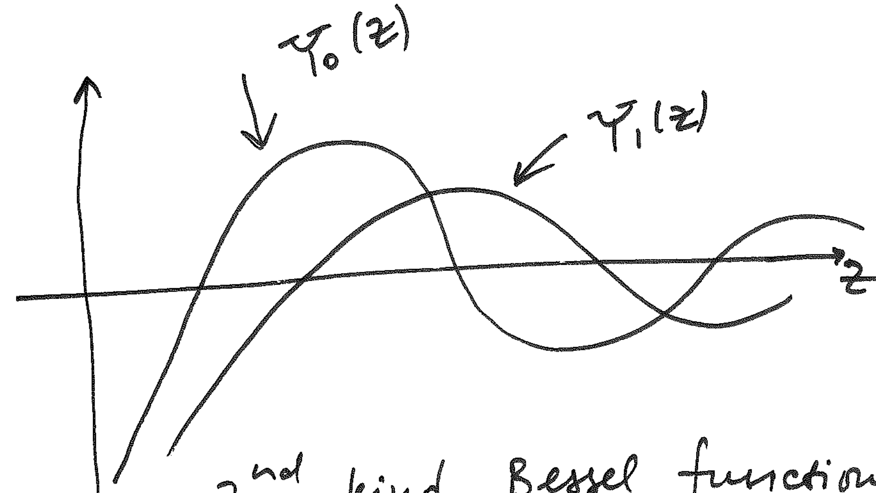
In our case, we take $m=1$

Roughly speaking, in Bessel equation $k \to 1$ and $c \to 0$ as $z \to \infty$

$\therefore$ $f(z)$ should oscillate like sin or cos as $z \to \infty$ and decay very slowly.



1st kind Bessel functions



2nd kind Bessel functions

$$J_m(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - \frac{m\pi}{2}\right) \quad \text{as } z \to \infty$$

$$Y_m(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - \frac{m\pi}{2}\right) \quad \text{as } z \to \infty$$

<u>Note</u>: $\sqrt{\frac{2}{\pi z}}$ decays very slowly as $z \to \infty$, much slower than any exponential.