University of Idaho

Lecture 45

Large eigenvalues

The eigenvalues $\sqrt{\lambda_{mn}}$ are proportional to the zeros of $J_m(z)$:

$$\sqrt{\lambda_{mn}} = \frac{z_{mn}}{a}$$

approximate zeros of $J_m(z)$: $z_{mn} - \frac{\pi}{4} - \frac{m\pi}{2} = \underbrace{-\frac{\pi}{2} + n\pi}_{\text{zeros of cosine}}$

$$\therefore \quad \boxed{z_{mn} \sim \pi\left(n - \frac{1}{4} + \frac{m}{2}\right)} \qquad \boxed{z_{m,n} - z_{m,n-1} \sim \pi}$$

Ex: $m=0$, z_{0n}

n	Exact	Approx	Error	% Error
1	2.40483	2.35619	0.04864	2%
2	5.52008	5.49779	0.02229	.4%
3	8.65373	8.63938	0.01435	.2%
4	11.79153	11.78097	0.01156	.1%

Normal modes

The solution for the vibrating membrane is a superposition of normal modes:

$$u_{mn}(r,\theta,t) = J_m\left(\frac{z_{mn}}{a} r\right) \begin{Bmatrix} \sin(m\theta) \\ \cos(m\theta) \end{Bmatrix} \begin{Bmatrix} \sin\left(c\frac{z_{mn}}{a} t\right) \\ \cos\left(c\frac{z_{mn}}{a} t\right) \end{Bmatrix}$$

When $m=0$, there are 2 families of solutions

When $m\neq 0$, there are 4 families of solutions

Nodal curves: curves along which a given normal mode vanishes for all times:

$$J_m\left(\frac{z_{mn}}{a} r\right)\sin m\theta = 0 \qquad \text{or} \qquad J_m\left(\frac{z_{mn}}{a} r\right)\cos m\theta = 0$$

See plot on pg. 315.

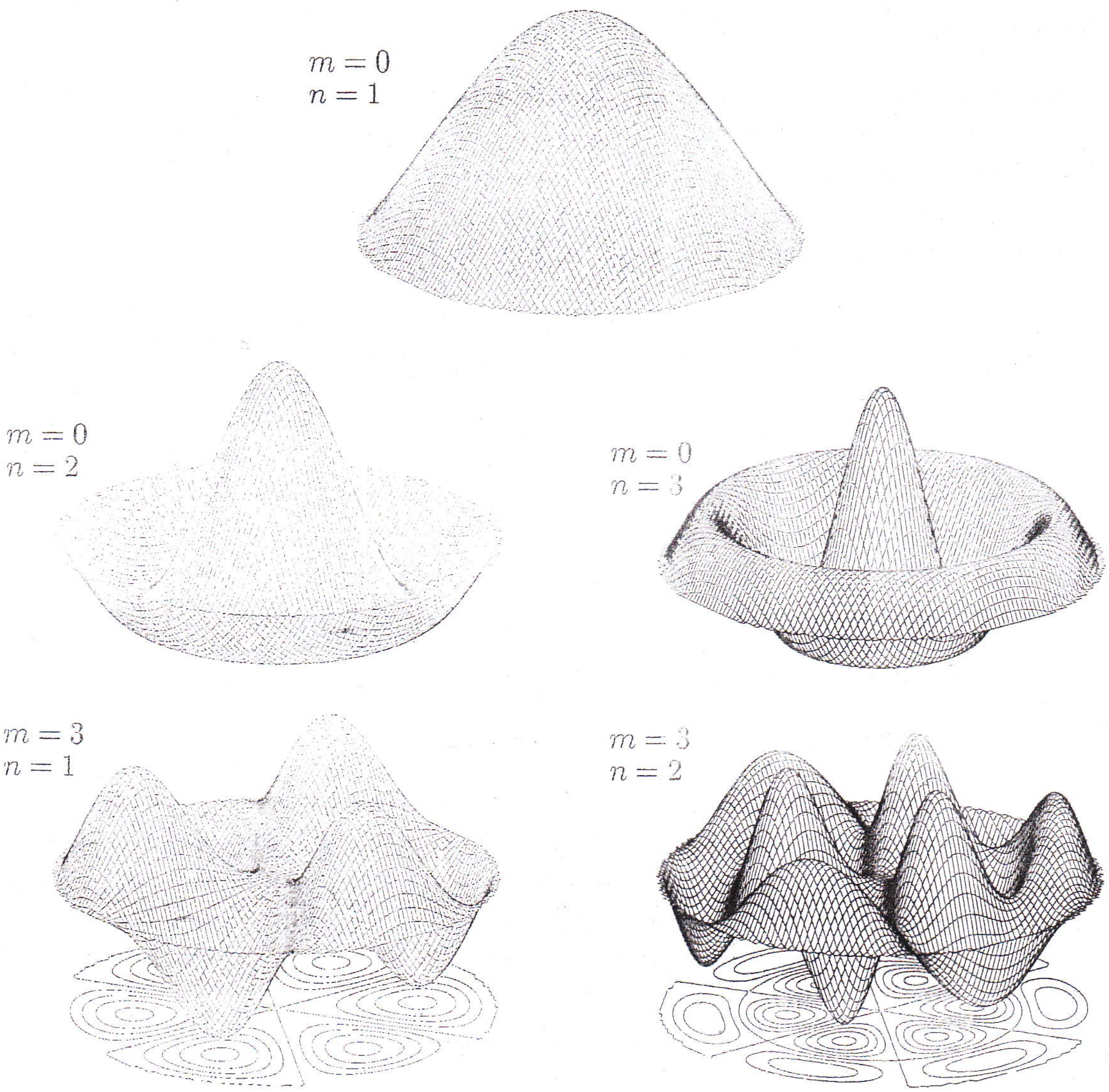


FIGURE 7.8.3 Normal nodes and nodal curves for a vibrating circular membrane.

nown as a **normal mode of oscillation** and is graphed for fixed t in Fig. 7.8.3. For each $\neq 0$, there are four families of solutions (for $m = 0$, there are two families). Each mode scillates with a characteristic natural frequency, $c\sqrt{\lambda_{mn}}$. At certain positions along the embrane, known as **nodal curves**, the membrane will be unperturbed for all time (for brating strings, we called these positions nodes). The nodal curve for the $\sin m\theta$ mode determined by

$$J_m\left(z_{mn}\frac{r}{a}\right)\sin m\theta = 0. \tag{7.8.6}$$

he nodal curve consists of all points where $\sin m\theta = 0$ or $J_m(z_{mn}r/a) = 0$; $\sin m\theta$ is

3

Nonhomogeneous Problems

General Problem

$u_t = K u_{xx} + Q(x,t)$	$u_{tt} = c^2 u_{xx} + Q(x,t)$	$u_{xx} + u_{yy} = Q(x,y)$
$u(0,t) = A(t)$	$u(0,t) = A(t)$	$u(x,0) = f_1(x)$
$u(L,t) = B(t)$	$u(L,t) = B(t)$	$u(x,H) = f_2(x)$
$u(x,0) = f(x)$	$u(x,0) = f(x)$	$u(0,y) = g_1(y)$
	$u_t(x,0) = g(x)$	$u(L,y) = g_2(y)$
Heat equation	Wave equation	Poisson equation

I Time independent BCs / No source

$u_t = K u_{xx}$ w/ $u(0,t) = A$, $u(L,t) = B$

$u(x,0) = f(x)$

Equilibrium Temp: $\lim_{t\to\infty} u(x,t) = w(x)$

$\Rightarrow w'' = 0$ w/ $w(0) = A$, $w(L) = B$ $\Rightarrow$ $\boxed{w(x) = A + \frac{B-A}{L} x}$

Displacement from equilibrium: $u(x,t) = v(x,t) + w(x)$

$\Rightarrow v_t = k v_{xx}$ w/ $v(0,t) = v(L,t) = 0$

$$v(x,0) = f(x) - \left[A + \frac{B-A}{L} x\right] \equiv g(x)$$

$$\therefore \quad v(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$a_n = \frac{2}{L} \int_0^L g(x) \cdot \sin \frac{n\pi x}{L} \, dx$$

II Steady source

$u_t = k u_{xx} + Q(x)$ w/ $u(0,t) = A$, $u(L,t) = B$

$u(x,0) = f(x)$

Equilibrium Temp: $w'' = -\frac{1}{k} Q(x)$, $w(0) = A$, $w(L) = B$

$$w' = c_1 - \frac{1}{k} \int_0^x Q(\xi) \, d\xi$$

$$w(x) = C_1 x + C_2 - \frac{1}{K}\int_0^x \int_0^{\xi} Q(\eta)\,d\eta\,d\xi$$

$$w(0) = A \Rightarrow C_2 = A$$

$$w(L) = B \Rightarrow C_1 L + A - \frac{1}{K}\int_0^L \int_0^{\xi} Q(\eta)\,d\eta\,d\xi = B$$

$$\therefore \boxed{w(x) = \frac{x}{L}\left[B - A + \frac{1}{K}\int_0^L \int_0^{\xi} Q(\eta)\,d\eta\,d\xi\right] + A - \frac{1}{K}\int_0^x \int_0^{\xi} Q(\eta)\,d\eta\,d\xi}$$

$$v_t = K v_{xx} \quad w/ \quad v(0,t) = v(L,t) = 0$$

$$v(x,0) = f(x) - w(x) \equiv g(x)$$

III General Problem

$$u_t = K u_{xx} + Q(x,t) \qquad u(x,0) = f(x)$$

$$u(0,t) = A(t) \quad \text{and} \quad u(L,t) = B(t)$$

In general, no equilibrium solution exists

Can always make BCs homogeneous by defining ANY reference temp. distribution, $r(x,t)$, that satisfies the BCs:

$$r(0,t) = A(t), \qquad r(L,t) = B(t)$$

Simplest choice: $\quad r(x,t) = A(t) + \frac{x}{L}\left[B(t) - A(t)\right]$

Let $u(x,t) = v(x,t) + r(x,t)$

$$\boxed{\begin{array}{ll} v_t = k\, v_{xx} + \bar{Q}(x,t) & v(x,0) = f(x) - r(x,0) \\ v(0,t) = v(L,t) = 0 & \\ \text{where } \quad \bar{Q}(x,t) = Q(x,t) - r_t + k\, r_{xx} & \end{array}}$$

Method of Eigenfunction Expansions

- Works for general source terms
- must have Homogeneous BCs

$$v_t = \mathcal{L}(v) + \bar{Q}(x,t) \qquad v(x,0) = g(x)$$

$$v(0,t) = v(L,t) = 0$$

$\mathcal{L}$ contains only spatial derivatives

Step 1 : solve problem w/ $\bar{Q}(x,t)=0$

$$v(x,t)=\sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \text{where} \qquad \begin{array}{l} \mathcal{L}(\phi_n)+\lambda_n \phi_n=0 \\ \phi(0)=\phi(L)=0 \end{array}$$

Step 2 : let $a_n=a_n(t)$ and plug back into full eq^{n}

$$v(x,t)=\sum_{n=1}^{\infty} a_n(t)\,\phi_n(x) \qquad v_t=\sum_{n=1}^{\infty} a_n'(t)\,\phi_n(x) \qquad \mathcal{L}=\frac{\partial^2}{\partial x^2}$$

$$\mathcal{L}(v)=\sum_{n=1}^{\infty} a_n(t)\,\mathcal{L}(\phi_n(x))=-\sum_{n=1}^{\infty} a_n(t)\,\lambda_n\,\phi_n(x)\,k$$

let $\bar{Q}(x,t)=\sum_{n=1}^{\infty} \bar{q}_n(t)\,\phi_n(x)$ $\qquad \bar{q}_n(t)=\dfrac{\int_0^L \bar{Q}(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$

$$\Rightarrow \sum_{n=1}^{\infty}\left[a_n'(t)+k\lambda_n a_n(t)-\bar{q}_n(t)\right]\phi_n(x)=0$$

$\therefore \quad a_n'+k\lambda_n a_n=\bar{q}_n$: ODE for coefficients

@ $t=0$: $\quad v(x,0)=g(x)=\sum_{n=1}^{\infty} a_n(0)\,\phi_n(x) \qquad \therefore \quad a_n(0)=\dfrac{\int_0^L g(x)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$

$$e^{k\lambda_n t} a_n' + k\lambda_n e^{k\lambda_n t} a_n = e^{k\lambda_n t} \bar{q}_n$$

(Method of Integrating Factor)

$$\left[a_n e^{k\lambda_n t}\right]' = e^{k\lambda_n t} \bar{q}_n \quad \rightarrow \quad \text{integrate} \int_0^t$$

$y' + P(x) y = Q(x)$

$\rho = e^{\int P(x)dx}$

$y\rho = \int \rho Q dx + C$

$y = \frac{1}{\rho}\left[\int \rho Q dx + C\right]$

$$a_n(t) e^{k\lambda_n t} - a_n(0) = \int_0^t e^{k\lambda_n \tau} \bar{q}_n(\tau)\, d\tau$$

$$\therefore \quad \boxed{a_n(t) = e^{-k\lambda_n t} a_n(0) + e^{-k\lambda_n t} \int_0^t e^{k\lambda_n \tau} \bar{q}_n(\tau)\, d\tau}$$

<u>Note</u>: PDE, ICs are satisfied. BCs are satisfied because of $\phi_n(x)$

Ex $u_t = u_{xx} + \sin(3x)e^{-t}, \qquad u(x,0) = f(x)$

$u(0,t) = 0 \quad$ and $\quad u(\pi, t) = 1$

Make BCs homogeneous: $v(x,t) = u(x,t) - \frac{x}{\pi}$

$v_t = v_{xx} + \sin(3x)e^{-t}, \qquad v(0) = f(x) - \frac{x}{\pi}$

$v(0,t) = v(\pi, t) = 0$

$\therefore \quad v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx), \qquad \lambda_n = n^2$

$\underset{f(x)-\frac{x}{\pi}}{\underset{\parallel}{v(x,0)}} = \sum_{n=1}^{\infty} a_n(0)\sin(nx) \Rightarrow \qquad a_n(0) = \frac{2}{\pi}\int_0^{\pi}\left[f(x) - \frac{x}{\pi}\right]\sin(nx)\,dx$

$\bar{Q}(x,t) = \sin(3x)e^{-t} = \sum_{n=1}^{\infty}\bar{q}_n(t)\,\phi_n(x) = \sum_{n=1}^{\infty}\bar{q}_n(t)\cdot\sin(nx)$

$\Rightarrow \quad \bar{q}_n(t) = \frac{2}{\pi}\int_0^{\pi} e^{-t}\sin(3x)\sin(nx)\,dx = \begin{cases} 0 & \text{if } n \neq 3 \\ e^{-t} & \text{if } n = 3 \end{cases}$

$a_n(t) = a_n(0)e^{-n^2 t} + e^{-n^2 t}\int_0^t \bar{q}_n(\tau)e^{n^2\tau}\,d\tau$

$$a_n(t) = \begin{cases} a_n(0)\, e^{-n^2 t} & \text{if } n \neq 3 \\ a_3(0)\, e^{-9t} + e^{-9t} \int_0^t e^{-\tau} e^{9\tau}\, d\tau = \end{cases}$$

$$= a_3(0) e^{-9t} + e^{-9t} \int_0^t e^{8\tau}\, d\tau = a_3(0) e^{-9t} + \frac{e^{-9t}}{8} e^{8\tau} \Big|_0^t =$$

$$= a_3(0) e^{-9t} + \frac{1}{8} e^{-9t} \left(e^{8t} - 1\right) = a_3(0) e^{-9t} + \frac{1}{8}\left(e^{-t} - e^{-9t}\right)$$

$$\therefore\ u(x,t) = \frac{x}{\pi} + \frac{1}{8}\left(e^{-t} - e^{-9t}\right) + \sum_{n=1}^{\infty} a_n(0) \sin(nx)\, e^{-n^2 t}$$

combine terms w/ $n \neq 3$ and $n = 3$

<u>Note</u>: first make BC homogeneous, then compute $a_n(t)$

More on Source Terms

Consider the heat eq^{n} w/ nonhomogeneous BCs

$u_t = k u_{xx} + Q(x,t) \qquad u(x,0) = f(x)$

$u(0,t) = A(t)$ and $u(L,t) = B(t)$

If we ignore the source and pretend that the BCs are homogeneous, then the separation of variables yields

$$\phi_n''(x) + \lambda_n \phi_n(x) = 0 \quad \text{w/} \quad \phi_n(0) = \phi_n(L) = 0$$

assume
$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x)$$

Note: Technically we should write $u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x)$ since at $x=0$ and $x=L$ the series converges to 0, not $A(t)$ and $B(t)$.

$$\Rightarrow u_t = \sum_{n=1}^{\infty} b_n'(t)\,\phi_n(x)$$

let $Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\,\phi_n(x)$ where $q_n(t) = \dfrac{\int_0^L Q(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$

$$u_t = k u_{xx} + Q(x,t)$$

$$\Rightarrow \sum_{n=1}^{\infty} b_n'(t)\,\phi_n(x) = k\,u_{xx} + \sum_{n=1}^{\infty} q_n(t)\,\phi_n(x)$$

Orthogonality: and linear independence $\quad b_n'(t) = \dfrac{\int_0^L [k u_{xx}]\,\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} + q_n(t)$

$$\int_0^L u_{xx}\,\phi_n(x)\,dx \overset{\text{by}}{\underset{\text{parts}}{=}} \left| \begin{array}{ll} \alpha = \phi_n & d\beta = u_{xx}\,dx \\ d\alpha = \phi_n'\,dx & \beta = u_x \end{array} \right| =$$

$$= \Big[\cancel{\phi_n(x)}^{\,0}\, u_x(x,t)\Big]_0^L - \int_0^L \phi_n'\,u_x\,dx \overset{\text{by}}{\underset{\text{parts}}{=}} \left| \begin{array}{ll} \alpha = \phi_n' & d\beta = u_x\,dx \\ d\alpha = \phi_n''\,dx & \beta = u \end{array} \right| =$$

$$= -\left[u\,\phi_n'\right]_0^L + \int_0^L u\,\underbrace{\phi_n''}_{= -\lambda_n \phi_n}\,dx \;=$$

$\phi_n'' + \lambda \phi_n = 0 \qquad \phi_n(0) = \phi_n(L) \quad \Rightarrow \quad \phi_n(x) = \sin\frac{n\pi x}{L}$

$\phi_n' = \frac{n\pi}{L} \cos\frac{n\pi x}{L}$

$\circleddash \left[A(t) - (-1)^n B(t)\right] \cdot \frac{n\pi}{L} - \lambda_n \int_0^L u(x,t)\, \phi_n(x)\, dx$

but $\quad b_n(t) = \dfrac{\int_0^L u(x,t)\, \phi_n(x)\, dx}{\int_0^L \phi_n^2(x)\, dx}$

$\therefore$

$$\boxed{\begin{aligned} & b_n'(t) + K\lambda_n b_n(t) = g_n(t) + \frac{K\left(\frac{n\pi}{L}\right)\left[A(t) - (-1)^n B(t)\right]}{\int_0^L \phi_n^2(x)\, dx} \\ & \text{w/} \quad b_n(0) = \frac{\int_0^L f(x)\, \phi_n(x)\, dx}{\int_0^L \phi_n^2(x)\, dx} \end{aligned}} \qquad (*)$$

Ex $\quad u_t = u_{xx} + \sin(3x)e^{-t}, \qquad u(0,t)=0, \quad u(\pi,t)=1$

We showed previously (see pg 11 of this lecture)

$$\boxed{u(x,t) = \frac{x}{\pi} + \frac{1}{8}\left(e^{-t} - e^{9t}\right) + \sum_{n=1}^{\infty} a_n(0)\sin(nx)e^{-n^2t}}$$

New approach (*): $\quad u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(nx)$

$$b_n' + n^2 b_n = q_n + 2n[0-(-1)^n] \quad \Big| \cdot e^{n^2t}$$

$$q_n = \begin{cases} 0 & \text{if } n\neq 3 \\ e^{-t} & \text{if } n=3 \end{cases} = \delta_{n3}e^{-t} \qquad \delta_{n3} = \begin{cases} 0 & \text{if } n\neq 3 \\ 1 & \text{if } n=3 \end{cases}$$

Kronecker delta

$$\Rightarrow \left[e^{n^2t}b_n\right]' = \delta_{n3}e^{(n^2-1)t} - 2n(-1)^n e^{n^2t}$$

Integrate $\int_0^t$:

$$e^{n^2t}b_n(t) - b_n(0) = \delta_{n3}\frac{1}{n^2-1}\left[e^{(n^2-1)t} - 1\right] - \frac{2}{n}(-1)^n\left[e^{n^2t} - 1\right]$$

$$\therefore\ b_n(t) = b_n(0)e^{-n^2t} + \frac{\delta_{n3}}{n^2-1}\left[e^{-t} - e^{-n^2t}\right] + \frac{2}{n}(-1)^n\left[e^{-n^2t} - 1\right]$$

$$\therefore\ \boxed{u(x,t) = \frac{1}{8}\left[e^{-t} - e^{-9t}\right] + \sum_{n=1}^{\infty}\left[\left(b_n(0) + \frac{2}{n}(-1)^n\right)e^{-n^2t} - \frac{2}{n}(-1)^n\right]\sin(nx)}$$

Q Which approach is better?

In general, the first approach (make BCs homogeneous) converges faster. In other words, if you truncate series in both solutions, the first solution will be more accurate.

Q Why?

In the second approach, Gibbs phenomena is necessarily present because $\phi_n(x)=0$ at the boundaries but the solution $u(x,t) \neq 0$ at the boundaries.

Poisson's Equation (Laplace's eq^{n} w/ source)

$\nabla^2 u = Q$ on domain Ω

$u = \alpha$ on boundary $\partial\Omega$

Problem is nonhomogeneous in 2 ways:

1. nonhomogeneous eq^{n} (source term)
2. nonhomog. BCs.

Let $u = u_1 + u_2$

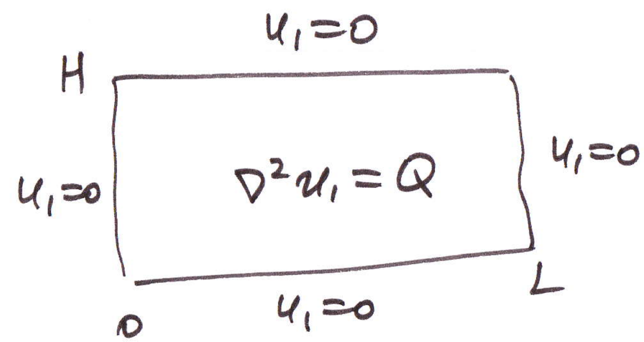
Problem #1: $\nabla^2 u_1 = Q$

$u_1 = 0$ on $\partial\Omega$

Problem #2: $\nabla^2 u_2 = 0$

$u_2 = \alpha$ on $\partial\Omega$

We already know how to solve Problem #2 (Laplace's eq^{n}) using separation of variables.



Approach #1 1D eigenfunctions

$\nabla^2 u_1 = Q$ w/ $u_1=0$ on $\partial\Omega$

Take 1D eigenfunctions that satisfy BCs.

Can choose either $\sin\frac{n\pi x}{L}$ or $\sin\frac{m\pi y}{H}$

Let $u_1(x,y) = \sum_{n=1}^{\infty} b_n(y)\sin\frac{n\pi x}{L}$ or $u_1(x,y) = \sum_{n=1}^{\infty} C_n(x)\sin\frac{m\pi y}{H}$

$$\nabla^2 u_1 = \sum_{n=1}^{\infty}\left[b_n''(y) - \left(\frac{n\pi}{L}\right)^2 b_n(y)\right]\sin\frac{n\pi x}{L} = Q(x,y)$$

Let $Q(x,y) = \sum_{n=1}^{\infty} q_n(y)\sin\frac{n\pi x}{L}$

$$\therefore\ b_n'' - \left(\frac{n\pi}{L}\right)^2 b_n = q_n \qquad \text{w/ } b_n(0) = b_n(H) = 0$$

This is a 2nd order ODE, nonhomog. Can solve by variation of parameters

$$b_n(y) = \sinh\left(\frac{n\pi(H-y)}{L}\right)\int_0^y q_n(\xi)\sinh\left(\frac{n\pi\xi}{L}\right)d\xi \; +$$

$$+ \sinh\left(\frac{n\pi y}{L}\right)\int_y^H q_n(\xi)\sinh\left(\frac{n\pi(H-\xi)}{L}\right)d\xi$$

$$\therefore \; u_1(x,y) = \sum_{n=1}^{\infty} b_n(y)\sin\left(\frac{n\pi x}{L}\right)$$

Approach #2 : 2D eigenfunctions

$$\nabla^2 u_1 = Q \quad \text{w/} \quad u_1 = 0 \;\text{ on } \partial\Omega$$

Consider the problem: $\nabla^2\phi = -\lambda\phi$ w/ $\phi = 0$ on $\partial\Omega$

From Section 7.3: $\phi_{nm}(x,y) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2$$

Let $u_1(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$

$$\nabla^2 u_1 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\nabla^2\phi_{nm} = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\lambda_{nm} b_{nm}\phi_{nm} = 0$$

By orthogonality:

$$b_{nm} = -\frac{1}{\lambda_{nm}} \frac{4}{LH} \int_0^H \int_0^L Q(x,y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dx\, dy$$

Note:

1. Approach #1 converges faster because it is only a single sum, not a double sum.

2. In general, approach #2 is easier because all we need is to apply orthogonality. Approach #1 requires the solution of a nonhomogeneous ODE.

Approach #2: for nonhomogeneous BCs

$\nabla^2 u = Q$ w/ $u = \alpha$ on $\partial\Omega$

Step 1: solve $\nabla^2 \phi = -\lambda \phi$ w/ $\phi = 0$ on $\partial\Omega$

Step 2: Let $u(x,y) = \sum_n \sum_m b_{nm} \phi_{nm}(x,y)$

Orthogonality:

$$b_{nm} = -\frac{1}{\lambda_{nm}} \frac{\iint u \nabla^2 \phi_{nm}\, dx\, dy}{\iint \phi_{nm}^2\, dx\, dy}$$

Integration by parts:

$$\iint u \nabla^2 \phi_{nm}\, dx\, dy = \iint \phi_{nm} \nabla^2 u\, dx\, dy + \oint (u \nabla \phi_{nm} - \phi_{nm} \nabla u) \cdot \vec{n}\, dS$$

but $\nabla^2 u = Q$

$$\therefore \quad b_n = -\frac{1}{\lambda_{nm}} \frac{\iint \phi_{nm} Q\, dx\, dy + \oint \alpha \nabla \phi_{nm} \cdot \vec{n}\, dS}{\iint \phi_{nm}^2\, dx\, dy}$$