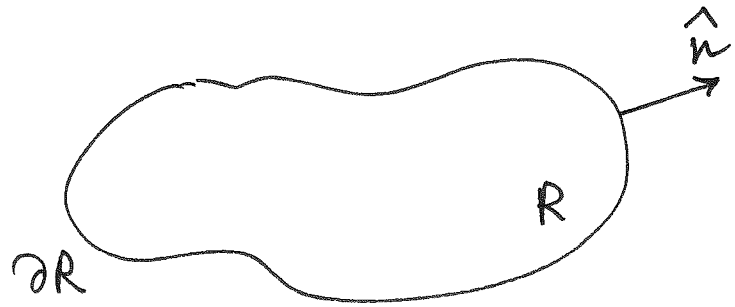
Lecture 4

1.5 Heat Equation in Higher Dimensions

Consider a region R in 3D



R: region

∂R: boundary of R

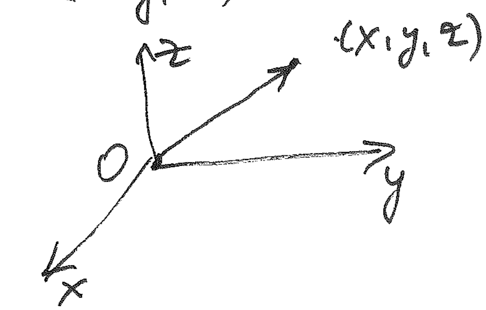
$\hat{n}$: outward unit normal to the boundary

The total thermal energy in R is

$$\iiint_R c(\vec{x})\,\rho(\vec{x})\,u(\vec{x},t)\,d\bar{V}$$

where $dV = dx\,dy\,dz$

$\vec{x} = (x, y, z)$

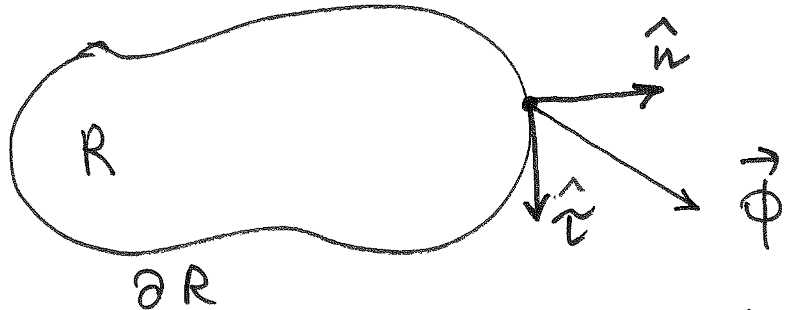


Conservation of thermal energy:

rate of change of total thermal energy	=	heat energy flowing across the boundary per unit time	+	energy generated inside region R due to sources/sinks per unit time

$$\text{rate of change of total thermal energy} = \frac{d}{dt} \iiint_R c(\vec{x})\,\rho(\vec{x})\, \underbrace{u(\vec{x},t)}_{\text{temperature}}\, dV$$

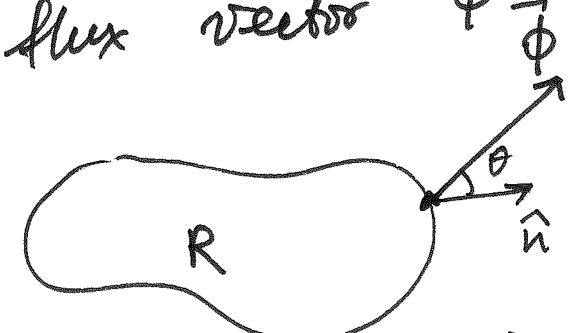
Let $\vec{\phi}$ be a heat flux vector that represents heat energy that flows per unit time per unit area. The magnitude of $\vec{\phi}$ is the thermal energy that flows across the boundary.



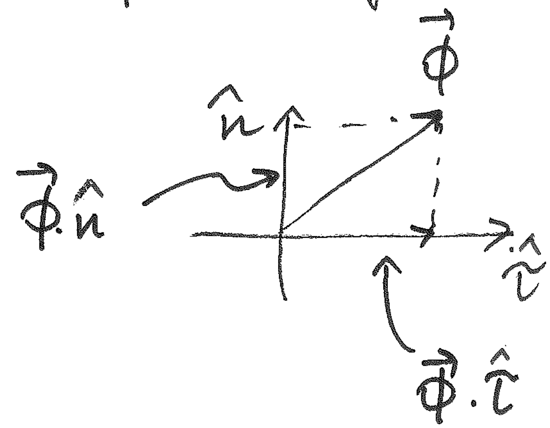
$\hat{n}$: outward unit normal

$\hat{\tau}$: unit tangent vector

Tangent component of $\vec{\Phi}$, i.e. $\vec{\Phi}\cdot\hat{\tau}$ has no contribution to the amount of energy that goes through the boundary. For $\hat{\tau}$, there is nothing that cross the boundary. Only normal component $\vec{\Phi}\cdot\hat{n}$ of heat flux vector $\vec{\Phi}$ contributes.



θ: angle between $\vec{\Phi}$ and $\hat{n}$



Component of $\vec{\Phi}$ in the normal direction $\hat{n}$ is

$$\vec{\Phi}\cdot\hat{n} = |\vec{\Phi}|\cdot\underbrace{|\hat{n}|}_{1}\cdot\cos\theta = |\vec{\Phi}|\cdot\cos\theta$$

Amount of thermal energy that crosses the boundary ∂R is

$$\oiint_{\partial R} \vec{\phi}\cdot\hat{n}\, dS$$

Heat flux is said to be <u>positive</u> if heat flows in the direction of outward unit normal $\hat{n}$.

<u>In</u> the conservation of energy equation, if heat flows out the region R, the total energy decreases

$\Rightarrow$

amount of energy that flows across the boundary $= - \oiint_{\partial R} \vec{\phi}\cdot\hat{n}\, dS$

The thermal energy due to sources/sinks is

$$\iiint_R Q(\vec{x})\, d\bar{V}$$

Conservation of energy:

$$\underbrace{\frac{d}{dt}\iiint_R c(\vec{x})\rho(\vec{x})\, u(\vec{x},t)\, d\bar{V}}_{=\ \iiint_R c(x)\rho(\vec{x})\frac{\partial u(\vec{x},t)}{\partial t}\, d\bar{V}} = -\iint_{\partial R} \vec{\phi}\cdot\hat{n}\, dS + \iiint_R Q(\vec{x})\, d\bar{V}$$

Gauss divergence Thm

Let $\vec{A} = \langle A_1, A_2, A_3 \rangle = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ be continuously differentiable. Then

$$\iiint_R \underbrace{\nabla\cdot\vec{A}}_{\operatorname{div}\vec{A}}\, dV = \oiint_{\partial R} \vec{A}\cdot\hat{n}\, dS$$

divergence of $\vec{A}$

where $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$: gradient vector

$\nabla u = \langle u_x, u_y, u_z \rangle$: gradient of u

$$\operatorname{div}\vec{A} = \nabla\cdot\vec{A} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle A_1, A_2, A_3 \rangle = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Using divergence Thm, we can write

$$\iint_{\partial R} \vec{\Phi} \cdot \hat{n} \, dS = \iiint_R \nabla \cdot \vec{\Phi} \, dV$$

Note: in 1D we had

$$\phi(a) - \phi(b) = - \int_a^b \frac{\partial \Phi}{\partial x} \, dx$$

Then the conservation of energy equation becomes

$$\iiint_R c\rho \frac{\partial u}{\partial t} \, dV = - \iiint_R \nabla \cdot \vec{\Phi} \, dV + \iiint_R Q \, dV$$

OR

$$\iiint_R \left[c\rho \frac{\partial u}{\partial t} + \nabla \cdot \vec{\Phi} - Q \right] dV = 0$$

Since R was an arbitrary region in $\mathbb{R}^3$ $\Rightarrow$

$$\boxed{c\rho \frac{\partial u}{\partial t} + \nabla \cdot \vec{\phi} - Q = 0}$$

conservation of energy in differential form

or

$$\boxed{c\rho \frac{\partial u}{\partial t} + \nabla \cdot \vec{\phi} = Q} \qquad (*)$$

Fourier's Law of Heat Conduction:

$$\vec{\phi} = -K_0(\vec{x}) \nabla u$$

where $K_0(\vec{x})$ is heat conduction coefficient.

Note: in 1D: $\phi = -K_0(x) \frac{\partial u}{\partial x}$

Here

$$\nabla u = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle = \frac{\partial u}{\partial x} \cdot \hat{i} + \frac{\partial u}{\partial y} \cdot \hat{j} + \frac{\partial u}{\partial z} \cdot \hat{k}$$

Now substitute expression for $\vec{\phi}$ in $(*)$

$$\boxed{c\rho \frac{\partial u}{\partial t} - \nabla \cdot (K_0 \nabla u) = Q}$$

the general heat equation

If material has uniform thermal properties, then c, ρ, K_0 are constant. Let set $Q = 0$, then we can write heat equation as

$$\frac{\partial u}{\partial t} = K \nabla^2 u$$

where $K = \frac{K_0}{c\rho}$: <u>thermal diffusivity coefficient</u>

$$\nabla^2 u = \nabla \cdot (\nabla u) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle =$$

$\equiv$

Δu

Laplacian of u

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) =$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

To solve the heat equation, we need to specify an initial condition and boundary conditions.

<u>Initial condition</u>: $u(x,y,z,0) = f(x,y,z)$: initial temperature

where $(x,y,z) \in R$

<u>Boundary conditions</u> :

1. <u>Prescribed temperature</u>,

$$u(x,y,z,t) = T(x,y,z,t), \quad \text{where } (x,y,z) \in \partial R$$

2. <u>Given flux</u>:

$$-K_0 \nabla u \cdot \hat{n} = \Phi(x,y,z,t), \quad (x,y,z) \in \partial R$$

For insulated boundary

$$\nabla u \cdot \hat{n} = 0 \quad \text{on } (x,y,z) \in \partial R$$

<u>Note</u>: $\nabla u \cdot \hat{n} = \frac{\partial u}{\partial n}$: directional derivative of u in direction of <u>unit</u> vector $\hat{n}$