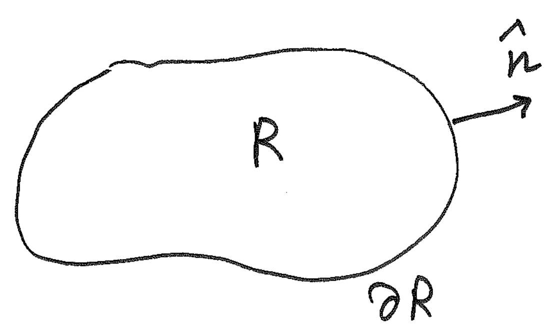
Lecture 4

1.5 Heat Equation in Higher Dimensions

Consider a region R in 3D space



R: region
∂R: boundary of R
$\hat{n}$: outward unit normal to boundary

The total thermal energy in R is

$$\iiint_R c(\vec{x})\,\rho(\vec{x})\,u(\vec{x},t)\,dV \qquad \text{where } dV = dx\,dy\,dz$$

Conservation of energy:

rate of change of total thermal energy = heat energy flowing across the boundary per unit time + energy generated inside region due to sources/sinks per unit time

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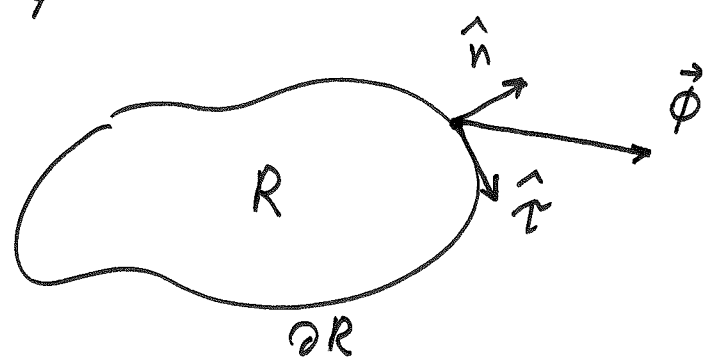
rate of change of total thermal energy $= \dfrac{d}{dt} \iiint\limits_R c(\vec{x})\rho(\vec{x})\, u(\vec{x},t)\, dV$

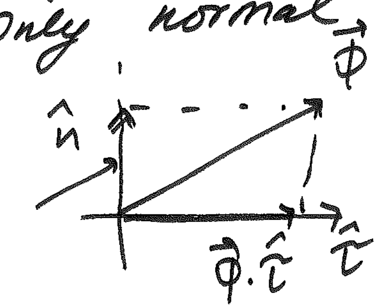
u: temperature

Let $\vec{\phi}$ be a heat flux vector that represents heat energy that flows per unit time per unit surface area. Magnitude of $\vec{\phi}$ is the thermal energy that flows across the boundary.



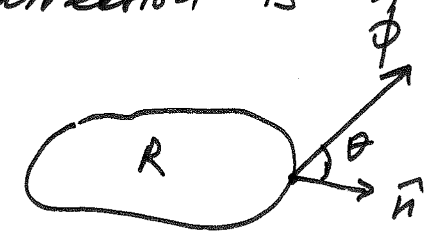
$\hat{\tau}$: unit tangent vector to ∂R

Tangent component of $\vec{\phi}$, i.e. $\vec{\phi}\cdot\hat{\tau}$ has no contribution to the amount of energy that goes across the boundary. For $\hat{\tau}$, there is nothing that crosses the boundary. Only normal component of $\vec{\phi}$ contributes: $\vec{\phi}\cdot\hat{n}$



Component of $\vec{\phi}$ in the normal direction is $\vec{\phi}\cdot\hat{n}$

$$\vec{\phi}\cdot\hat{n} = |\vec{\phi}|\cdot\underbrace{|\hat{n}|}_{=1}\cdot\cos\theta$$



$$\vec{\phi}\cdot\hat{n} = |\vec{\phi}|\cdot\cos\theta$$

Amount of thermal energy that goes across the Boundary ∂R is

$$\oint\!\!\oint_{\partial R} \vec{\phi}\cdot\hat{n}\, dS$$

Heat flux is positive if heat flows in the direction of outward unit normal $\hat{n}$. In the conservation of energy equation, if heat flows out the region R, the total energy decreases $\Rightarrow$

$$\text{amount of energy that flows across the boundary} = -\oint\!\!\oint_{\partial R} \vec{\phi}\cdot\hat{n}\, dS$$

Thermal energy due to sources/sinks is

$$\iiint_R Q(\vec{x})\, dV$$

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Conservation of energy:

rate of change of total energy = heat energy across the boundary per unit time + energy generated inside due to sources/sinks per unit time

$$\underbrace{\frac{d}{dt}\iiint_R c(\vec{x})\rho(\vec{x})\,u(\vec{x},t)\,d\bar{V}}_{\iiint_R c(\vec{x})\rho(\vec{x})\frac{\partial u(\vec{x},t)}{\partial t}\,d\bar{V}} = -\iint_{\partial R} \vec{\Phi}\cdot\hat{n}\,dS + \iiint_R Q(\vec{x})\,d\bar{V}$$

Gauss divergence thm

Let $\vec{A} = \langle A_1, A_2, A_3\rangle = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be continuously differentiable, then

$$\iiint_R \underbrace{\nabla\cdot\vec{A}}_{\text{div}\,\vec{A}}\,d\bar{V} = \oint\!\!\oint_{\partial R} \vec{A}\cdot\hat{n}\,dS$$

divergence of $\vec{A}$

where

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \text{: gradient operator}$$

$$\operatorname{div}\vec{A} = \nabla\cdot\vec{A} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle A_1, A_2, A_3 \rangle =$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Using divergence Thm, we can write

$$\iint_{\partial R} \vec{\phi}\cdot\hat{n}\, dS = \iiint_R \nabla\cdot\vec{\phi}\, dV$$

<u>Note</u>: in 1D we had $\phi(a) - \phi(b) = -\int_a^b \frac{\partial \phi}{\partial x}\, dx$

Then the energy conservation law becomes

$$\iiint_R c\rho \frac{\partial u}{\partial t}\, dV = -\iiint_R \nabla\cdot\vec{\phi}\, dV + \iiint_R Q\, dV$$

or $$\iiint_R \left[c\rho \frac{\partial u}{\partial t} + \nabla \cdot \vec{\Phi} - Q \right] d\bar{V} = 0$$

Since R was an arbitrary region in R^3 $\Rightarrow$

$$c\rho \frac{\partial u}{\partial t} + \nabla \cdot \vec{\Phi} - Q = 0$$

or $$\boxed{c\rho \frac{\partial u}{\partial t} = -\nabla \cdot \vec{\Phi} + Q} \qquad (*)$$

<u>Fourier's Law of Heat Conduction</u>:

$$\vec{\Phi} = -K_0(\vec{x})\, \nabla u$$

where $K_0(\vec{x})$ is heat conduction coefficient

<u>Note</u>: in 1D: $\quad \phi = -K_0(x) \dfrac{\partial u}{\partial x}$

and

$$\nabla u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}$$

Substitute expression for $\vec{\Phi}$ into eq^{n} $(*)$:

$$\boxed{c\rho\frac{\partial u}{\partial t} = \nabla\cdot(K_0 \nabla u) + Q}$$ the general heat equation

$K_0 = K_0(\vec{x})$

If material has uniform thermal properties, then c, ρ, K_0 are constant and if we can set $Q=0$, then we can rewrite this equation as

$$\frac{\partial u}{\partial t} = k\nabla^2 u \quad \text{where} \quad k = \frac{K_0}{c\rho} : \text{thermal diffusivity}$$

$$\nabla^2 u = \nabla\cdot(\nabla u) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right\rangle =$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \equiv \Delta u : \text{Laplacian of } u$$

To solve the heat equation, we need to identify an initial condition and boundary conditions.

- <u>Initial condition</u>: $u(x,y,z,0) = f(x,y,z)$: initial temperature

  where $(x,y,z) \in R$

- <u>Boundary conditions</u>:

  1. <u>Prescribed temperature</u>:

     $u(x,y,z,t) = T(x,y,z,t)$ where $(x,y,z) \in \partial R$ (on the boundary!)

  2. <u>Given heat flux</u>:

     $-K_0 \nabla u \cdot \hat{n} = \Phi(x,y,z,t), \quad (x,y,z) \in \partial R$

For insulated boundary,

$$\nabla u \cdot \vec{n} = 0 \qquad (x, y, z) \in \partial R$$

<u>Note</u>: $\nabla u \cdot \vec{n} = \dfrac{\partial u}{\partial n}$: directional derivative of u in the direction $\vec{n}$

3. <u>Newton's law of Cooling</u>

$$-K_0 \nabla u \cdot \vec{n} = H(u - u_B), \qquad (x, y, z) \in \partial R$$

u_B: ~~temperature~~ of surrounding medium

$H > 0$: convection coefficient