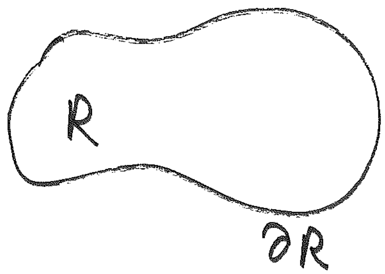
Lecture 5

Boundary conditions (Cont'd)

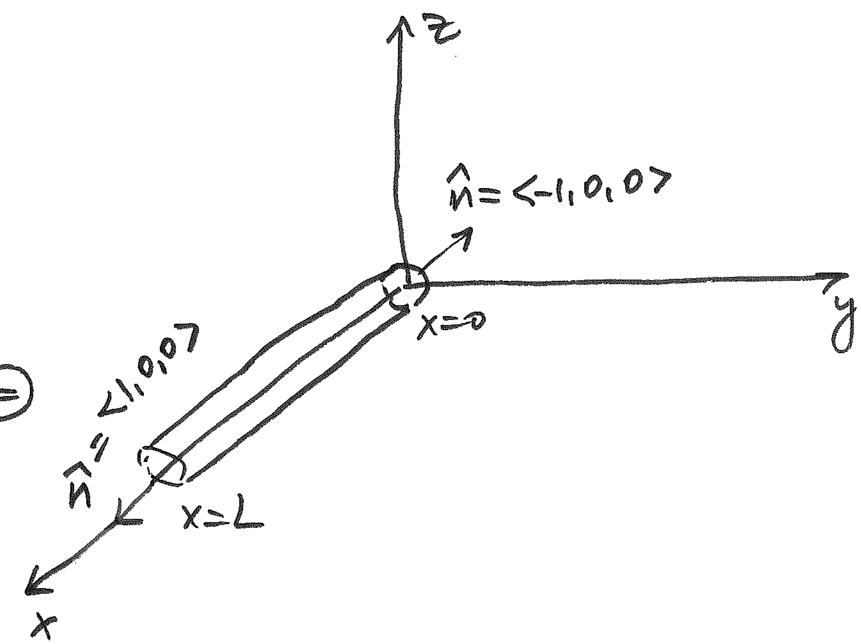
3. Newton's Law of Cooling

$$-K_0 \nabla u \cdot \hat{n} = H(u - u_B) \qquad (x, y, z) \in \partial R$$



u_B: temperature of surrounding medium

H: convection coefficient



Partial case: 1D

at $x=0$, $\hat{n} = \langle -1, 0, 0 \rangle$

$$-K_0 \nabla u \cdot \hat{n} = -K_0 \cdot \frac{\partial u}{\partial x}(-1) = K_0 \frac{\partial u}{\partial x}$$

$\nabla u = \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \rangle$, $\hat{n} = \langle -1, 0, 0 \rangle$

$$\therefore \quad K_0 \frac{\partial u}{\partial x}\Big|_{x=0} = H(u(0,t) - u_B)$$

or

$$\boxed{K_0 \frac{\partial u(0,t)}{\partial x} = H(u(0,t) - u_B)}$$

at $x = L$, $\hat{n} = \langle 1, 0, 0 \rangle$

$$-K_0 \nabla u \cdot \hat{n} = -K_0 \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle \cdot \langle 1, 0, 0 \rangle = -K_0 \frac{\partial u}{\partial x} \cdot 1$$

$$\therefore \quad \boxed{-K_0 \frac{\partial u(L,t)}{\partial x} = H(u(L,t) - u_B)}$$

Compare these BCs at $x=0$ & $x=L$ with those 1D versions we wrote in Lecture 3 (1/14/2019)

Steady-state

$$c\rho \underbrace{\cancel{\frac{\partial u}{\partial t}}}_{0} = \nabla\cdot(K_0 \nabla u) + Q$$

$$\Rightarrow \nabla\cdot(K_0 \nabla u) + Q = 0$$

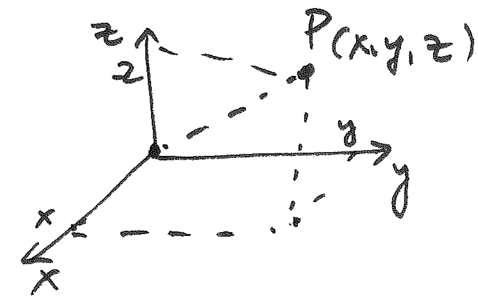
If $K_0 = \text{const} \Rightarrow$ $\boxed{\nabla^2 u = -\frac{Q}{K_0}}$ Poisson equation

If in addition $Q=0 \Rightarrow$

$\boxed{\nabla^2 u = 0}$ Laplace equation

Laplacian in Cartesian coordinates (x, y, z)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$



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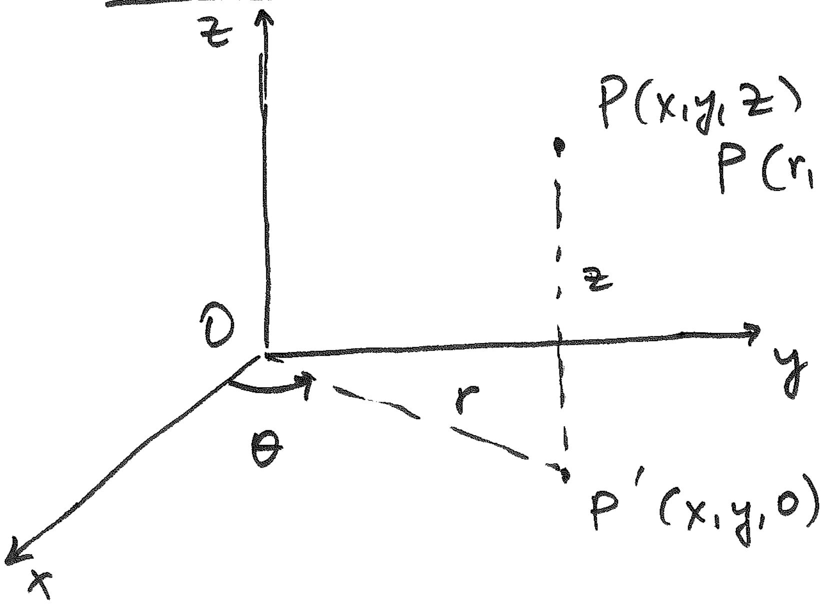
Laplacian in Cylindrical Coordinates



$0 \le \theta \le 2\pi, \quad r \ge 0$

$x = r\cos\theta$

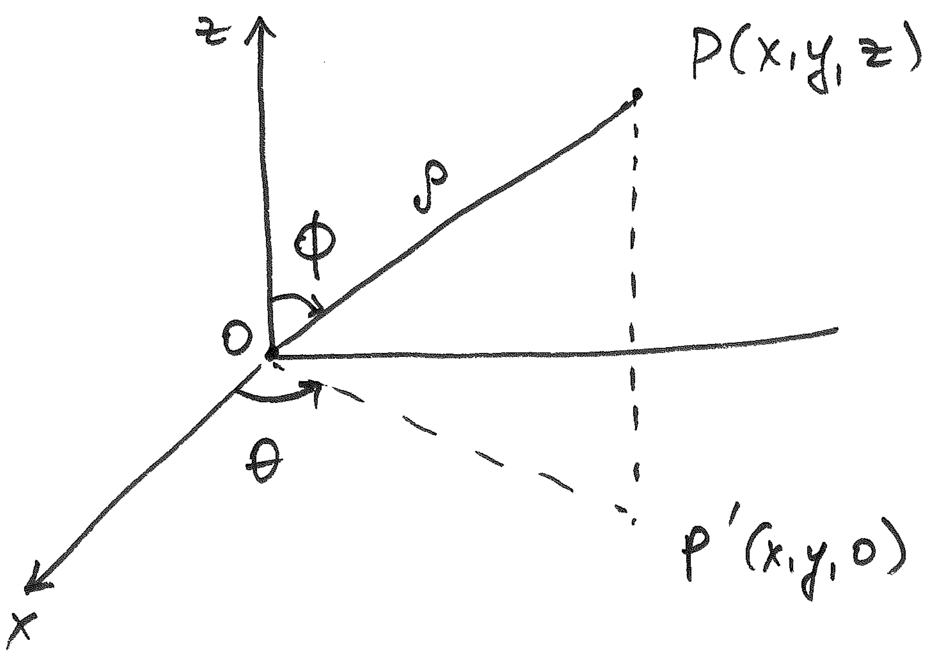
$y = r\sin\theta$

$z = z$

$r = \sqrt{x^2+y^2} \qquad \tan\theta = \frac{y}{x}$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

Laplacian in Spherical Coordinates



$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$\rho \ge 0$

$x = \rho \cos\theta \sin\phi$

$y = \rho \sin\theta \sin\phi$

$z = \rho \cos\phi$

$$\nabla^2 u = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 \sin\phi} \frac{\partial}{\partial \phi} \left(\sin\phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{\rho^2 \sin\phi} \frac{\partial^2 u}{\partial \theta^2}$$

Linearity

Consider a differential equation that can be written in operator form

$$\mathcal{L}u = f$$

DE is linear if operator $\mathcal{L}$ is linear.

Def Operator $\mathcal{L}$ is linear if

$$\mathcal{L}\{c_1 u_1 + c_2 u_2\} = c_1 \mathcal{L}\{u_1\} + c_2 \mathcal{L}\{u_2\}$$

where c_1, c_2 are arbitrary constants and u_1, u_2 are arbitrary functions.

Ex $\mathcal{L} = \dfrac{\partial}{\partial t}$ is a linear differential operator

$$\mathcal{L}\{c_1 u_1 + c_2 u_2\} = \frac{\partial}{\partial t}(c_1 u_1 + c_2 u_2) = c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial u_2}{\partial t}$$

differentiation is a linear operator

$$= c_1 \mathcal{L}\{u_1\} + c_2 \mathcal{L}\{u_2\} \quad \blacksquare$$

Ex $\mathcal{L} = \dfrac{\partial^2}{\partial x^2}$ is a linear differential operator

$$\mathcal{L}\{c_1 u_1 + c_2 u_2\} = \frac{\partial^2}{\partial x^2}\{c_1 u_1 + c_2 u_2\} = c_1 \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial^2 u_2}{\partial x^2}$$

$$= c_1 \mathcal{L}\{u_1\} + c_2 \mathcal{L}\{u_2\} \quad \blacksquare$$

Claim Any linear combination of linear operators is a linear operator.

Pf HW

Consider a partial case with two linear operators. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two linear operators.

Let
$$\mathcal{L} = a_1 \mathcal{L}_1 + a_2 \mathcal{L}_2$$
$$\mathcal{L}\{u\} = a_1 \mathcal{L}_1\{u\} + a_2 \mathcal{L}_2\{u\} \qquad (1)$$

Goal: to show that $\mathcal{L}$ is a linear operator, i.e.
$$\mathcal{L}\{c_1 u_1 + c_2 u_2\} \overset{?}{=} c_1 \mathcal{L}\{u_1\} + c_2 \mathcal{L}\{u_2\}$$

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Indeed,

$$\mathcal{L}\{c_1 u_1 + c_2 u_2\} \overset{(1)}{=} a_1 \mathcal{L}_1\{c_1 u_1 + c_2 u_2\} + a_2 \mathcal{L}_2\{c_1 u_1 + c_2 u_2\}$$

$$\underset{\text{are linear operators}}{\overset{\mathcal{L}_1,\mathcal{L}_2}{=}} a_1\left[c_1 \mathcal{L}_1\{u_1\} + c_2 \mathcal{L}_1\{u_2\}\right] + a_2\left[c_1 \mathcal{L}_2\{u_1\} + c_2 \mathcal{L}_2\{u_2\}\right] =$$

$$\underset{\text{terms}}{\overset{\text{re-group}}{=}} c_1\left[a_1 \mathcal{L}_1\{u_1\} + a_2 \mathcal{L}_2\{u_1\}\right] + c_2\left[a_1 \mathcal{L}_1\{u_2\} + a_2 \mathcal{L}_2\{u_2\}\right] =$$

$$\overset{(1)}{=} c_1 \mathcal{L}\{u_1\} + c_2 \mathcal{L}\{u_2\}$$

Hence, $\mathcal{L}$ is a linear operator.

Example of the proof using the method of mathematical induction.

Claim: $1+2+\ldots+n=\dfrac{n(n+1)}{2}$

Step 1: verify that the statement is true for $n=1$ (or $n=2$)

$n=1$: $1=\dfrac{1\cdot 2}{2}$ ✓ $\qquad$ $n=2$: $1+2=\dfrac{2(2+1)}{2}$ ✓

Step 2: Induction assumption: assume that the formula is true for $n=k$, ie.

$$1+2+\ldots+k=\frac{k(k+1)}{2}$$

Step 3: Induction step: show that the formula is true for $n=k+1$ using the induction assumption.

We need to show that

$$1+2+\ldots+k+(k+1)\overset{?}{=}\frac{(k+1)((k+1)+1)}{2}=\frac{(k+1)(k+2)}{2}$$

$$\underbrace{1+2+\ldots+k}_{\frac{k(k+1)}{2} \text{ by induction assumption}}+(k+1)=\frac{k(k+1)}{2}+(k+1)=(k+1)\left(\frac{k}{2}+1\right)=(k+1)\frac{k+2}{2}=\frac{(k+1)((k+1)+1)}{2}$$

■

Since $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$ are linear operators, the "heat operator" $\frac{\partial}{\partial t} - K\frac{\partial^2}{\partial x^2}$ is also a linear operator.

$$\Rightarrow \quad \frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + Q \quad \text{or} \quad \left(\frac{\partial}{\partial t} - K\frac{\partial^2}{\partial x^2}\right)u = Q$$

is a linear differential equation.

<u>Def</u> <u>linear</u> DE $\mathcal{L}u = f$ is <u>homogeneous</u> if $f \equiv 0$. Otherwise, DE is <u>nonhomogeneous</u>.