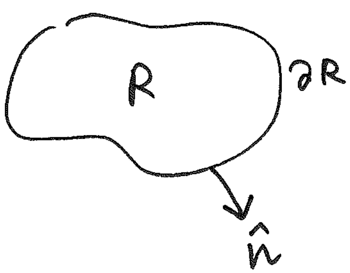
University of Idaho

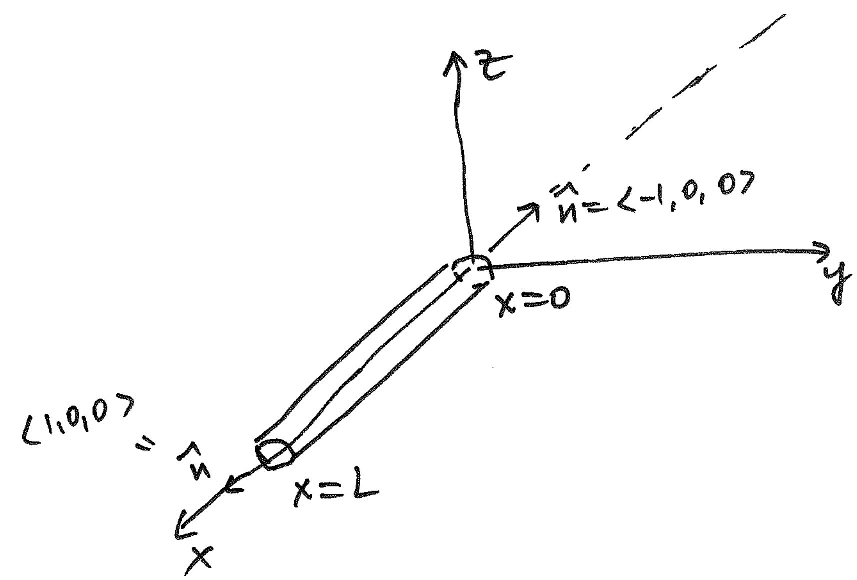
Lecture 5

BC: <u>Newton's law of Cooling</u>

$$\underbrace{-K_0 \nabla u \cdot \hat{n}}_{\substack{\text{heat flux}\\ \text{in direction}\\ \hat{n}}} = H(u - u_B), \qquad (x,y,z) \in \partial R$$



<u>Partial case</u>: 1D



at $x=0$, $\hat{n} = \langle -1, 0, 0 \rangle = -\hat{\imath}$

$$-K_0 \nabla u \cdot \hat{n} = -K_0 \cdot \frac{\partial u}{\partial x} \cdot (-1) = K_0 \frac{\partial u}{\partial x}$$

$\nabla u = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle$, $\hat{n} = \langle -1, 0, 0 \rangle$

$$\Rightarrow \boxed{K_0(0) \frac{\partial u}{\partial x}(0,t) = H\left(u(0,t) - u_B\right)}$$

at $x = L$, $\quad \hat{n} = \langle 1, 0, 0 \rangle = \hat{i}$

$$-K_0 \nabla u \cdot \hat{n} = -K_0 \frac{\partial u}{\partial x} \cdot 1 = -K_0 \frac{\partial u}{\partial x}$$

$$\Rightarrow \boxed{-K_0(L) \frac{\partial u}{\partial x}(L,t) = H\left(u(L,t) - u_B\right)}$$

Compare these BCs at $x=0$ and $x=L$ with those we discussed in Lecture 3 (1/20/2017).

<u>Steady-state</u>

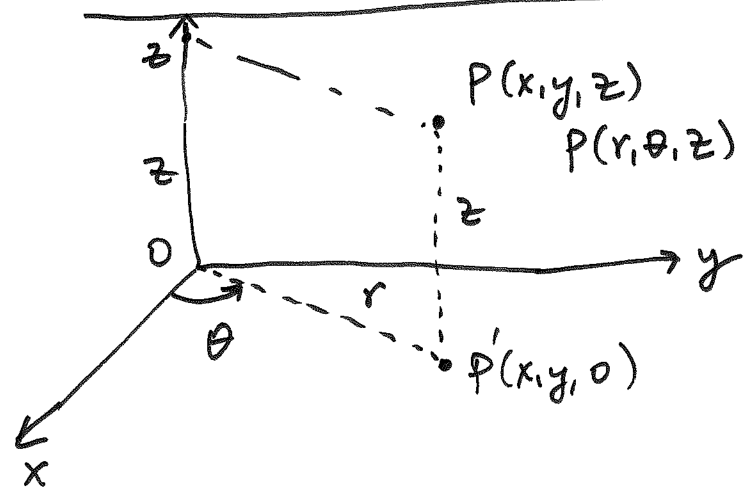
$$c\rho \cancelto{0}{\frac{\partial u}{\partial t}} = \nabla \cdot (K_0 \nabla u) + Q$$

$$\Rightarrow \nabla \cdot (K_0 \nabla u) + Q = 0$$

If $K_0 = \text{const} \Rightarrow \boxed{\nabla^2 u = -\frac{Q}{K_0}}$ <u>Poisson equation</u>

If in addition $Q = 0 \Rightarrow \boxed{\nabla^2 u = 0}$ <u>Laplace equation</u>

Laplacian in Cylindrical Coordinates



$0 \le \theta < 2\pi \quad , \quad r \geq 0$

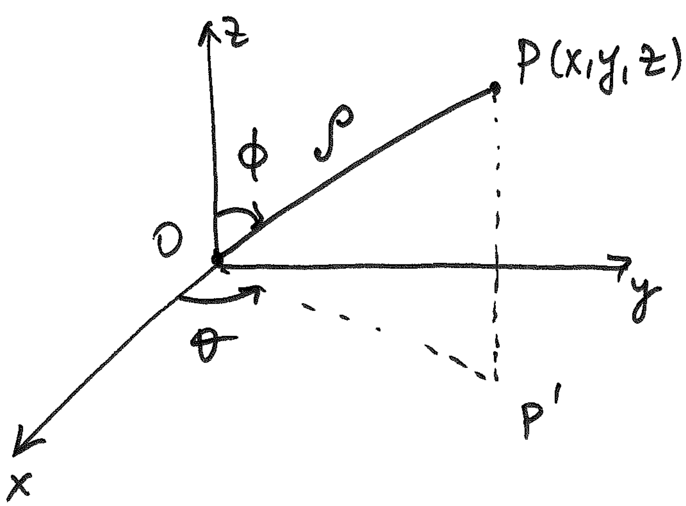
$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

Laplacian in Spherical Coordinates



$0 \le \theta < 2\pi$

$0 \le \phi \le \pi$

$x = \rho \cos\theta \sin\phi$

$y = \rho \sin\theta \sin\phi$

$z = \rho \cos\phi$

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial^2 u}{\partial \theta^2}$$

Linearity

Consider a differential equation that can be written as

$$\mathcal{L}u = f$$

DE is linear if operator $\mathcal{L}$ is linear.

<u>Def</u> Operator $\mathcal{L}$ is <u>linear</u> if

$$\mathcal{L}\{c_1 u_1 + c_2 u_2\} = c_1 \mathcal{L}\{u_1\} + c_2 \mathcal{L}\{u_2\}$$

where c_1, c_2 are arbitrary constants and u_1, u_2 are arbitrary functions.

<u>Ex</u> $\mathcal{L} = \dfrac{\partial}{\partial t}$ is a linear differential operator.

$$\mathcal{L}\{c_1 u_1 + c_2 u_2\} = \frac{\partial}{\partial t}(c_1 u_1 + c_2 u_2) \overset{\text{differentiation}}{=}$$

is a linear operation

$$= c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial u_2}{\partial t} = c_1 \mathcal{L}\{u_1\} + c_2 \mathcal{L}\{u_2\} \quad \blacksquare$$

<u>Ex</u> $\mathcal{L} = \dfrac{\partial^2}{\partial x^2}$ is a linear differential operator.

$$\mathcal{L}\{c_1 u_1 + c_2 u_2\} = \frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2) = c_1 \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial^2 u_2}{\partial x^2} =$$

$$= c_1 \mathcal{L}\{u_1\} + c_2 \mathcal{L}\{u_2\} \quad \blacksquare$$

__Claim__ Any linear combination of linear operators is a linear operator.

__Pf__ HW

Consider a partial case with two linear operators.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two linear operators.

Let

$$\mathcal{L} = a_1 \mathcal{L}_1 + a_2 \mathcal{L}_2$$

$$\mathcal{L}\{u\} = a_1 \mathcal{L}_1\{u\} + a_2 \mathcal{L}_2\{u\} \qquad (1)$$

__Goal__: show that $\mathcal{L}$ is a linear operator, i.e.

$$\mathcal{L}\{c_1 u_1 + c_2 u_2\} \overset{?}{=} c_1 \mathcal{L}\{u_1\} + c_2 \mathcal{L}\{u_2\}$$

Indeed,

$$\mathcal{L}\{c_1 u_1 + c_2 u_2\} \overset{(1)}{=} a_1 \mathcal{L}_1\{c_1 u_1 + c_2 u_2\} + a_2 \mathcal{L}_2\{c_1 u_1 + c_2 u_2\} \underset{\substack{\text{are}\\\text{linear}\\\text{operators}}}{\overset{\mathcal{L}_1, \mathcal{L}_2}{=}}$$

$$= a_1\left[c_1 \mathcal{L}_1\{u_1\} + c_2 \mathcal{L}_1\{u_2\}\right] + a_2\left[c_1 \mathcal{L}_2\{u_1\} + c_2 \mathcal{L}_2\{u_2\}\right] \underset{\text{terms}}{\overset{\text{re-group}}{=}}$$

$$= c_1\left[a_1 \mathcal{L}_1\{u_1\} + a_2 \mathcal{L}_2\{u_1\}\right] + c_2\left[a_1 \mathcal{L}_1\{u_2\} + a_2 \mathcal{L}_2\{u_2\}\right] =$$

$$= c_1 \mathcal{L}\{u_1\} + c_2 \mathcal{L}\{u_2\}$$

Hence, $\mathcal{L}$ is a linear operator. ■

Example of the proof using the method of mathematical induction.

Claim: $1+2+\ldots+n = \frac{n(n+1)}{2}$

Step 1: verify that the statement is true for $n=1$ (or $n=2$)

$n=1$: $1 = \frac{1\cdot 2}{2}$ ✓ $\qquad n=2$: $1+2 = \frac{2(2+1)}{2}$ ✓

Step 2: Induction assumption: assume that the formula is true for $n=k$, ie.

$$1+2+\ldots+k = \frac{k(k+1)}{2}$$

Step 3: Induction step: show that the formula is true for $n=k+1$ using the induction assumption.

We need to show that

$$1+2+\ldots+k+(k+1) \stackrel{?}{=} \frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$\underbrace{1+2+\ldots+k}_{\frac{k(k+1)}{2} \text{ by induction assumption}}+(k+1) = \frac{k(k+1)}{2} + (k+1) = (k+1)\left(\frac{k}{2}+1\right) = (k+1)\,\frac{k+2}{2} = \frac{(k+1)((k+1)+1)}{2}$$

■

Since $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$ are linear operators, the "heat operator" $\frac{\partial}{\partial t} - K\frac{\partial^2}{\partial x^2}$ is also a linear operator

$$\Rightarrow \quad \frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + Q \qquad \text{or} \qquad \underbrace{\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2}}_{\left(\frac{\partial}{\partial t} - K\frac{\partial^2}{\partial x^2}\right)\{u\}} = Q$$

$$\left(\frac{\partial}{\partial t} - K\frac{\partial^2}{\partial x^2}\right)\{u\} = Q$$

is a linear differential equation.

<u>Def</u> Linear DE $\mathcal{L}u = f$ is <u>homogeneous</u> if $f = 0$. Otherwise, the linear DE is <u>nonhomogeneous</u> (i.e. $f \neq 0$).

<u>Principle of Linear Superposition</u>

If u_1 and u_2 satisfy the same linear homogeneous differential equation, then their linear combination $c_1 u_1 + c_2 u_2$ is also a solution of the same equation.

Pf Since u_1 and u_2 satisfy the same diff. equation, we can write

$$\mathcal{L}\{u_1\}=0 \quad \text{and} \quad \mathcal{L}\{u_2\}=0$$

Then

$$\mathcal{L}\{c_1u_1+c_2u_2\} \underset{\text{a linear operator}}{\overset{\mathcal{L}\text{ is}}{=}} c_1\cancelto{0}{\mathcal{L}\{u_1\}} + c_2\cancelto{0}{\mathcal{L}\{u_2\}} = 0$$

$\therefore$ $c_1u_1+c_2u_2$ satisfies the same $\underbrace{\text{linear}}$ $\overset{\text{homog.}}{\text{DE}}$.

Boundary Conditions

The concept of linearity and homogeneity also applies to BCs.

Examples of linear BCs:

$$\left.\begin{aligned} u(0,t) &= f(t) \\ u_x(L,t) &= g(t) \end{aligned}\right\} \quad \text{linear nonhomog. BCs}$$

$u_x(0,t)=0$ linear homog. BC.