Lecture 6

Principle of Linear Superposition

Recall, a linear DE $\mathcal{L}u = f$ is <u>homogeneous</u> if $f=0$, ie. $\mathcal{L}u = 0$.

If u_1 and u_2 satisfy the same <u>linear</u> homogeneous DE, then their linear combination $c_1 u_1 + c_2 u_2$ is also a solution of the same equation.

Pf

Since u_1, u_2 satisfy the same linear homog. DE, we can write

$\mathcal{L}\{u_1\} = 0$ and $\mathcal{L}\{u_2\} = 0$

We need to show that $\mathcal{L}\{c_1 u_1 + c_2 u_2\} = 0$.

<u>Indeed</u>,

$$\mathcal{L}\{c_1 u_1 + c_2 u_2\} \underset{\text{linear}}{\overset{\mathcal{L}\text{ is}}{=}} c_1 \cancelto{0}{\mathcal{L}\{u_1\}} + c_2 \cancelto{0}{\mathcal{L}\{u_2\}} = 0$$

$\therefore$ $c_1 u_1 + c_2 u_2$ satisfies the same linear homog. DE

Boundary Conditions

The concept of linearity and homogeneity also applies to BCs.

<u>Examples of linear BCs:</u>

$u(0,t) = f(t)$: linear nonhomog. BC

$u_x(L,t) = g(t)$: linear nonhomog. BC

$u_x(0,t) = 0$: linear homog. BC

$-K_0(L)\, u_x(L,t) = H[u(L,t) - g(t)]$: linear nonhomog. BC

Aside

$y = \underbrace{ax}_{\text{linear term}} + b$: linear function

$y = ax^2 + bx + c$: quadratic f^2

Nonlinear BC:

$$u_x(L,t) = \underbrace{u^2(L,t)}_{\text{nonlinear}}$$

Def a homogeneous (linear) BC is the condition satisfied by a trivial solution ($u \equiv 0$)

$e^{u(x,t)}$: nonlinear

$\sqrt{u(x,t)}$: nonlinear

$\ln u_x$: —"—

Chapter 2 Separation of Variables

Consider the linear homogeneous 1D heat equation with homogeneous BCs:

$$u_t = k u_{xx} \qquad 0 \le x \le L$$

$$u(0,t) = 0, \qquad u(L,t) = 0$$

$$u(x,0) = f(x)$$

Note To solve a nonhomogeneous problem we need to learn first how to solve a homog. problem.

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Separation of variables

Look for solutions of the form

$$u(x,t) = \phi(x)\, G(t)$$

Separation of variables, introduced by Bernoulli, allows one to reduce PDE to ODE.

Note Separation of variables can only be used on linear homogeneous DEs w/ linear homogeneous BCs.

$$\frac{\partial u}{\partial t} = \phi(x)\frac{dG}{dt} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{dx^2}\, G(t)$$

Substitute these derivatives in the heat eq^n

$$u_t = k\, u_{xx} :$$

$$\phi(x)\,\frac{dG}{dt} = k \cdot \frac{d^2\phi}{dx^2}\, G(t) \quad \Big|\ \frac{1}{k\,\phi(x)\,G(t)}$$

$$\underbrace{\frac{\frac{dG}{dt}}{k\,G(t)}}_{\substack{f^n \text{ of } t \\ \text{alone}}} = \underbrace{\frac{\frac{d^2\phi}{dx^2}}{\phi(x)}}_{\substack{f^n \text{ of } x \\ \text{alone}}} = \underbrace{-\lambda}_{\text{(separation constant)}}$$

i.e. both sides should be equal to some constant, called a sepation constant.

We will "ignore" IC for now.

We obtain two ODEs:

$$\frac{\frac{dG}{dt}}{k\,G(t)} = -\lambda \quad \text{or} \quad \boxed{\frac{dG}{dt} + \lambda k\, G(t) = 0}$$

time dependent problem for $G(t)$

$$\frac{\frac{d^2\phi}{dx^2}}{\phi(x)} = -\lambda \quad \text{or} \quad \boxed{\frac{d^2\phi}{dx^2} + \lambda \phi = 0}$$

x dependent problem for $\phi(x)$

The goal is to construct a nontrivial solution, i.e. nonzero solution.

Boundary conditions

$$u(0,t) = 0 \Rightarrow \phi(0)\underbrace{G(t)}_{\neq 0} = 0 \Rightarrow \boxed{\phi(0) = 0}$$

since we need a nontrivial solution

$u(L,t)=0 \Rightarrow \phi(L)\underbrace{G(t)}_{\neq 0}=0 \Rightarrow \boxed{\phi(L)=0}$

The time-dependent problem for $G(t)$:

$\frac{dG}{dt} + \lambda K\, G(t) = 0$: separable ODE

$G \equiv 0$ is also a sol$^{\underline{n}}$

$\frac{dG}{dt} = -\lambda K\, G(t)$

$\frac{dG}{G} = -\lambda K\, dt \qquad G \neq 0$

$\int \frac{dG}{G} = -\lambda K \int dt$

$\ln|G| = -\lambda K t + \tilde{C} \quad | \exp \qquad e^{\ln x} = x$

$$|G| = e^{-\lambda k t + \tilde{c}} = e^{-\lambda k t} \cdot e^{\tilde{c}} > 0$$

to combine w/ $G \equiv 0$ we can write

$$\boxed{G(t) = C e^{-\lambda k t}}$$

C can be >0, <0 and $=0$

C: arbitrary const

Boundary Value Problem for $\phi(x)$ (BVP)

$$\frac{d^2\phi}{dx^2} + \lambda \phi(x) = 0$$

$\phi(0) = 0$, $\phi(L) = 0$

This is called an <u>eigenvalue problem</u>.

λ is an <u>eigenvalue</u> and $\phi(x) \not\equiv 0$ is an <u>associated eigenfunction</u>.

<u>Def</u> λ is called an <u>eigenvalue</u> if there exists a nontrivial function $\phi(x)$, called an eigenfunction, that satisfies the above BVP.

We need to find all possible λ's for which we can find nontrivial $\phi(x)$

<u>Case I</u> <u>e'values and e'functions for $\lambda > 0$.</u>

$$\frac{d^2\phi}{dx^2} + \lambda \phi(x) = 0, \qquad \lambda > 0$$

$x'' + 4x = 0$

$x = e^{rx}$

$(D^2+4)x = 0$

assume $\phi(x) = e^{rx}$. $\Rightarrow$ $\phi' = re^{rx}$, $\phi'' = r^2 e^{rx}$

$r^2 e^{rx} + \lambda \cdot e^{rx} = 0$

$\underset{\neq 0}{e^{rx}} (r^2 + \lambda) = 0$ $\Rightarrow$ $r^2 + \lambda = 0$: characteristic eq^{n}

$r^2 = -\lambda$ $\quad \lambda > 0$

$r = \pm i\sqrt{\lambda}$

$\Rightarrow$ solutions are $e^{i\sqrt{\lambda}x}$, $e^{-i\sqrt{\lambda}x}$

$\quad\quad = \cos\sqrt{\lambda}x + i\sin\sqrt{\lambda}x$, $\quad = \cos\sqrt{\lambda}x - i\sin\sqrt{\lambda}x$

In fact, real & imaginary parts, $\cos\sqrt{\lambda}x$, $\sin\sqrt{\lambda}x$ are also solutions $\Rightarrow$

$$\boxed{\phi(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x}$$

Operator approach:

$(D^2 + \lambda)\phi = 0$

$\pm i\sqrt{\lambda}$

$\phi = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$\phi(0)=0 \Rightarrow c_1 \underbrace{\cos 0}_{=1} + c_2 \cancel{\sin 0}^{\,0} = 0 \Rightarrow \boxed{c_1=0}$

$\phi(x) = c_2 \sin \sqrt{\lambda} x$

$\phi(L)=0 \Rightarrow \underbrace{c_2}_{\neq 0} \sin(\sqrt{\lambda} L) = 0 \Rightarrow \sin(\sqrt{\lambda} L) = 0$

for a nontrivial solution

$\sqrt{\lambda} L = n\pi, \quad n = 1, 2, 3, \ldots$

$$\Rightarrow \boxed{\begin{array}{ll} \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots & \text{e'values} \\ \phi_n(x) = \sin \sqrt{\lambda_n} x = \sin \frac{n\pi x}{L} : & \text{e'functions} \end{array}}$$